Price Discrimination and Privacy: A Note

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Abstract

In many instances of price discrimination, a seller of an item is in possession of signals from competing buyers regarding their private valuation for the item. While intuition may suggest that a seller should always exploit all available information, our paper shows that it is actually in the interests of the seller to strategically ignore the information contained in the signals with positive probability, i.e. to (at least) sometimes price the item in a non-discriminatory way according to a fixed rule. Such a "mixed" strategy induces buyers to send more informative signals in equilibrium. Thus the seller can offset any revenue loss in states where he ignores information by the gains made due to the larger amount of information now communicated in the signals.

1 Introduction

Textbook models of first-degree price discrimination are based on a seller making a take-it-or-leave-it price offer under the assumption of perfect information regarding buyers' preferences (for example, Varian 1989, Section 2.2). However, as noted by Varian (Section 2.2.2), the perfect information assumption is not likely to hold in the real world. Therefore, a seller will price an item based on some prior beliefs regarding the distribution of buyers' preferences. However, in between the textbook model based on perfect information, and pricing based on prior beliefs alone, lies an interesting class of examples that have not hitherto been analyzed in the literature in a systematic way. These instances of (first-degree) price discrimination are distinguished by the fact that a seller is in possession of additional information in the form of signals from the buyers reflecting their private valuation. Thus, while information is not perfect, it is also not as coarse as implied by some unconditional prior regarding the distribution of buyers' valuations. The seller is therefore in a position to extract information from these signals regarding each buyer's private valuation and then allocate the item to one of the buyers at a take-it-or-leave-it price. Some examples of this phenomenon include the following:

- The second-chance offer of eBay: The auction site eBay collects bids from potential buyers and then assigns the item to the highest bidder. Additional units of the item

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are then sold to the losing set of bidders at an estimate of their valuations (Salmon and Wilson 2006). Thus, after assigning the winning item, eBay uses the information contained in the bids (signals) to price discriminate among losing bidders.

- Resale in an open cry auction: The winner of an open cry auction is privy to the bids at which the losing bidders dropped out of the auction. The winner, armed with these signals, can use the information contained therein to resell the item to one of the losing bidders at a discriminatory price (Krishna 2002, Section 4.4).
- Phantom signaling in second price auctions: Price discrimination can take a more subtle form in second-price auctions as noted by Lucking-Reiley (2000). The seller collects all bids and assigns the item to the highest bidder. Since submitting a bid equal to private valuation is a weakly dominant strategy in second-price auctions, the bids are an accurate signal of each buyer's private valuation. The seller then price discriminates by interspersing a phantom second bidder with a fictitious valuation just below the winning bid. The winning bidder thus pays a price higher than the legitimate second-highest bid.

It seems intuitive that an optimizing seller should strategically exploit all available information. However, if buyers are cognizant that any signal they provide to the seller to determine a winner amongst them will actually be used to price discriminate to their detriment, then they will incorporate this information into their signaling behavior. Thus, in competitive situations where there is one item and many buyers, each buyer will confront the following trade-off: the dictates of competition will demand that any signal sent be an accurate reflection of the actual valuation; on the other hand, the proclivity of the seller to price discriminate will demand that signals be sufficiently noisy so that a seller cannot capture all the surplus from the exchange. This tension between information revelation and privacy raises a number of interesting issues that provide the main motivation for our paper:

1. How do buyers address the conflict between information revelation due to competition and the concern for privacy due to price discrimination? We prove that when ex-ante symmetric buyers have an incentive to preserve the privacy of their valuations, then the strategic interaction between buyers and the seller yields a symmetric Bayesian-Nash equilibrium in *non-invertible* strategies (Lemmas 1 and 2). Exploiting the commonality of our framework with that of cheap talk models, we characterize equilibria in terms of *partition* signaling strategies. In a partition strategy the valuation space of each buyer is partitioned into a finite number of intervals. A buyer chooses a random signal from an

appropriate signaling interval corresponding to the realized valuation. Thus the signal sent by the buyer to the seller is *noisy* in the sense of indicating the valuation interval but incapable of being "inverted" by the seller to reveal the true private valuation. The seller chooses the highest interval that is signaled and allocates the object to the buyer in this interval (choosing one randomly if there are multiple buyers). Such a partition equilibrium optimally resolves the trade-off faced by each buyer by revealing just enough information to signal the valuation interval and enough noise to hide the exact valuation.

2. How much information do buyers actually transmit to sellers? The amount of information that is communicated in equilibrium depends on coarseness or fineness of the partition strategy. In particular, if buyers expect that the seller will always price discriminate, then the signals are completely uninformative (Theorem 1). The buyers have no incentive to reveal any information at all to the seller. On the other hand, if buyers believe that, with some positive probability, the seller will not price discriminate but adhere to some fixed non-discriminatory pricing rule, then the partition strategy is finer and more information is revealed in equilibrium (Theorem 2).

3. The behavior of the buyers poses an important question for the seller: is it always optimal to price discriminate? Or, should the seller sometimes behave in a non-discriminatory manner by strategically ignoring information contained in the signals and committing to a fixed pricing rule? The seller is aware that price discrimination creates incentives for the buyers to protect privacy by employing noisy partition strategies. We show that it is strictly better for the seller to adhere to a fixed pricing rule with positive probability than it is to price discriminate with probability one (Proposition 1). As noted earlier, this induces the buyers to divulge more information. Therefore, the seller is able to offset any potential revenue loss in states where he ignores information and behaves non-discriminatorily by the gains made due to the greater informational content of signals.

A related contribution of our paper is to draw attention to the formal similarity between our model of price discrimination where buyers value privacy and the class of sender-receiver *cheap talk* games inspired by Crawford and Sobel (1982). Similar to cheap talk, the number and size of the partitions in the equilibrium signaling strategy are indicative of the information contained in the signals. These in turn are a function of the (common) prior beliefs of the buyers, given by the probability ϵ, regarding the proclivity of the seller to price discriminate (which in cheap talk translates into the divergence in the preferences of the sender and receiver). If $\epsilon = 1$, then we have a "babbling" equilibrium in which the signals contain no information. However if $0 < \epsilon < 1$, then the equilibrium partition is finer and buyers transmit more information about private valuations in their signals.

It may be noted that our model is closer to the *multiple-sender* version of cheap talk games which also addresses competition among senders when determining the extent of information to reveal (for example, Krishna and Morgan, 2001).[3] However there is an important difference in that the payoff function for senders in cheap talk is symmetric around the utility-maximizing action. This implies that each sender receives a non-zero payoff whether the action chosen by the receiver is above or below the desired action of the sender. This fact plays a critical role in determining the no-arbitrage conditions that characterize the dividing points of the partition equilibrium. In contrast, our exchange model has a natural *one-sided asymmetry*: a buyer only gets a non-zero payoff if the action (price) chosen by the seller is smaller than the buyer's valuation. This asymmetry complicates the construction of a partition equilibrium.

The paper is organized as follows. The model, and the description of partition equilibria, is provided in Section 2. The Bayesian-Nash equilibrium of the price discrimination game is presented in Section 3. Section 4 considers the case where a seller is allowed to behave non-discriminatorily. Section 5 concludes.

2 The Model

There are N ex-ante symmetric buyers for an object that is being offered by a seller. Each buyer is risk-neutral. Buyer i has a private valuation V_i for the object – the maximum amount that he is willing to pay – which is iid on the interval $[0, 1]$ according to a distribution function F with associated density f. Buyer i sends a signal b_i to the seller. Let the vector of signals be denoted by $\mathbf{b} = (b_1, b_2, ..., b_N)$, and b_{-i} denote the vector of signals from buyers other than i. Also let Δ denote the collection of probability distributions on the set of buyers. The seller assigns the object according to some function $Q(\mathbf{b}) \in \Delta$ where $Q_i(\mathbf{b})$ is the probability that buyer i wins the object. For example, if the seller assigns the object to the buyer with the highest signal (and randomly allocates to one of the highest signals in the event of a tie), then $Q_i(\mathbf{b}) = 1$ if $b_i > \max_{j \neq i} b_j$ and $Q_i(\mathbf{b}) = \frac{1}{K}$ if b_i is one of K highest signals. The seller chooses a price $p(b)$ as a function of the winning signal b. A buyer i with realized valuation

[3] There are also multiple sender games where the senders observe a multidimensional state variable (Battaglini, 2002). In contrast, each buyer in our model observes a unidimensional signal regarding own valuation.

v_i and signal b_i receives a utility of $v_i - p(b_i)$ from winning the object and zero otherwise. Therefore we can write the expected payoff of buyer i as:

$$u(b_i, v_i) = W(b_i)(v_i - p(b_i)) \tag{1}$$

where $W(b_i)$ is the probability of winning the object with a signal b_i. This probability will of course depend on the signaling strategies employed by the other buyers and the assignment rule Q employed by the seller.

A discriminating seller will attempt to extract all information from each signal and assign the object to the buyer who the seller believes has the highest valuation. The seller will then choose a price to maximize expected revenue. Let $h(v_i|b_i)$ denote the conditional expectation of V_i given b_i. If the seller assigns the object to buyer i, then the seller chooses a price p that maximizes:

$$R(p, b_i) = p \int_p^1 h(x|b_i)dx \tag{2}$$

All these facts are common knowledge among the players. We will refer to this game as the *price discrimination* (PD) game. We are now ready to define the equilibrium of this game. Given the ex-ante symmetry of the buyers, we restrict attention to symmetric equilibria.

Definition 1 *A symmetric equilibrium of the PD game is the signaling function* $\{\gamma(b|v) : v \in [0,1]\}$ *used by each buyer and a pricing strategy* $p(b)$ *for the seller such that:*

(A) *For each buyer* i *and any* $v_i \in [0,1]$, $\gamma(.|v_i)$ *has full support on some measurable subset* S_i *of* $[0,1]$. *Further:*

$$b_i^* \in \arg\max\{W(b_i)(v_i - p(b_i)) : b_i \in S_i\}$$

where $W(b_i)$ *is based on buyers other than* i *signaling according to* γ *and the seller using an assignment rule* Q.

(B) *Given the assignment rule* Q *and a winning signal* b, *the seller chooses a price* $p(b)$ *that maximizes* (2).

It is first useful to establish when buyers will adopt an invertible non-noisy signaling strategy in the Bayesian-Nash equilibrium. The first result shows that invertible signaling strategies are employed if sellers adhere to a fixed pricing rule, $\overline{p}$, that does not invert bids in order to capture larger surplus. Given that the seller is not reacting strategically to the buyer's choice of a signaling strategy, the symmetric Bayesian-Nash equilibrium of the fixed price game requires that buyers' strategy profile satisfies condition (A) in Definition 1. From now on we will let G represent the distribution function of Y, the second-highest of N valuations, and g represent the associated density function.

Lemma 1 *Suppose that for any vector* $\mathbf{b}$ *of signals,* $Q_i(b_i, b_{-i}) = 1$ *and* $\overline{p}(b_i) = b_i$ *if* $b_i > \max_{j \neq i} b_j$ *(the item is randomly allocated among winning buyers in the event of a tie). An invertible strategy* α *constitutes a symmetric Bayesian-Nash equilibrium if and only if:*

$$\alpha(v) = E[Y|Y < v] = \frac{1}{G(v)} \int_0^v y g(y) dy, \quad v > 0 \tag{3}$$

and $\alpha(0) = 0$.

Proof. We first prove necessity. Let α denote a monotonically increasing signaling strategy in a symmetric equilibrium. Clearly $\alpha(0) = 0$. Therefore, expected payoff is $G(\alpha^{-1}(b))[v-b]$. Maximizing this expression with respect to b and substituting $b = \alpha(v)$ in a symmetric equilibrium yields the differential equation:

$$\frac{d}{dv}[G(v)\alpha(v)] = vg(v)$$

Solving this differential equation using $\alpha(0) = 0$ yields (3). We next establish sufficiency. Suppose $N-1$ buyers employ strategy α except buyer i with valuation v who signals $\alpha(z)$ for $z \neq v$. The expected payoff from signaling $\alpha(z)$ when the true valuation is v is equal to:.

$$\begin{aligned} u(\alpha(z), v) &= G(z)[v - \alpha(z)] \\ &= G(z)v - \left[zG(z) - \int_0^z G(x)dx \right] \\ &= G(z)[v - z] + \int_0^z G(x)dx \end{aligned}$$

On the other hand, $u(\alpha(v), v) = \int_0^v G(x)dx$. It now follows that:

$$u(\alpha(v), v) - u(\alpha(z), v) = G(z)[z - v] - \int_v^z G(x)dx \geq 0, \quad z \neq v \tag{4}$$

Therefore buyer i does not have a profitable deviation from $\alpha(v)$. ■

We can now address the nature of the signaling strategy when the buyers know that the seller will price discriminate.

Lemma 2 *Suppose that for any vector* $\mathbf{b}$ *of signals,* $Q_i(b_i, b_{-i}) = 1$ *if* $b_i > \max_{j \neq i} b_j$ *(the item is randomly allocated among winning buyers in the event of a tie). Symmetric Bayesian-Nash equilibrium signaling strategies of the PD game cannot be invertible.*

Proof. Suppose $\alpha(v)$ is an invertible symmetric equilibrium strategy (corresponding to some degenerate $\gamma(.|v)$) and let b_i denote the signal of buyer i. Note that $h(v_i|b_i) = 1$ if $v_i = \alpha^{-1}(b_i)$ and 0 otherwise. A buyer i with valuation v_i who signals b_i wins the item with probability $\Pr[\max_{j \neq i} v_j \leq \alpha^{-1}(b_i)] = F(\alpha^{-1}(b_i))^{N-1} \equiv G(\alpha^{-1}(b_i))$. Therefore, if b_i is the winning signal, the seller's price from maximizing (2) is v_i. If there is more than one such signal, the seller randomly chooses one of the buyers with this signal. With strategy α each buyer's expected utility is zero. Now, holding the strategies of $N-1$ buyers and the seller fixed, consider a deviating buyer who uses the signaling strategy $\widehat{\alpha}(v) = \alpha\left(\frac{v}{2}\right)$. The probability of the buyer winning with strategy $\widehat{\alpha}$ is given by $G\left(\frac{v}{2}\right)$, which is non-zero when $v \neq 0$. The price paid is $\frac{v}{2}$, and the expected utility $\frac{v}{2} \times G\left(\frac{v}{2}\right) > 0$ when $v \neq 0$. Thus each buyer has an incentive to unilaterally deviate from α. ■

What kind of non-invertible strategies are candidates for equilibrium? We can now draw on the work of Crawford and Sobel (1982) on cheap talk games and consider equilibria comprising of *partition* strategies. These are strategies where buyers partition the valuation space into a finite number of disjoint intervals and submit an identical signal for valuations belonging to the same interval. It will be technically convenient in the subsequent analysis to work with a signaling space that is different from the valuation space. Accordingly, let β be any arbitrary monotonic increasing function satisfying $\beta(v) < v$ for all $v \in (0, 1]$ and $\beta(0) = 0$. Let $X(M) \equiv \{x_0, x_1, ...x_M\}$ denote a partition of the valuation space $[0, 1]$, $x_0 = 0$, $x_M = 1$, $x_{J-1} < x_J$, $J = 1, 2, ...M$.

Definition 2 *An* M*-partition strategy is a family of signaling functions* $\{\gamma(b|v) : v \in [0, 1]\}$ *such that* $\gamma(.|v)$ *is a uniform distribution on* $[\beta(x_{J-1}), \beta(x_J))$ *when* $v \in [x_{J-1}, x_J)$*,* $J = 1, 2, ..., M$*. With some abuse of notation, we will also let* $X(M)$ *denote the* M*-partition strategy.*

Remark: One may construe the partition strategy as an attempt by buyers to distinguish themselves from other (lower valuation) buyers while shielding their true valuation. The requirement that $\gamma(.|v)$ be a *uniform* distribution on the interval ensures, as noted in Gibbons (1992, pp.216-217), that there are no signals that are "out of equilibrium". We could equally well have chosen some other distribution for $\gamma(.|v)$. We could, for instance, have allowed

$\gamma(.|v)$ to be degenerate at some $b_J \in [\beta(x_{J-1}), \beta(x_J))$ when $v \in [x_{J-1}, x_J)$, where $b_1 < b_2 < \cdots < b_M$, $J = 1, 2, ..., M$. The seller's out-of-equilibrium beliefs in the degenerate case could then be specified as: for any $b \in [\beta(x_{J-1}), \beta(x_J))$, $b \neq b_J$, the valuations are distributed on $[x_{J-1}, x_J)$ according to F. Also note that the particular choice of the function β to define the signaling intervals does not limit the model in any way. Equilibria corresponding to different β are equivalent in the sense that they imply the same noisy signaling behavior.

3 The Discriminating Seller

The seller collects all signals and offers the object to the buyer from the highest interval. If there is more than one buyer in the highest interval, then one of them is selected randomly by the seller. Formally, if $[\beta(x_{J-1}), \beta(x_J))$ is the highest interval, and there are K signals in this interval, then $Q_i(\mathbf{b}) = \frac{1}{K}$ if $b_i \in [\beta(x_{J-1}), \beta(x_J))$. If the buyer refuses the seller's price, then the seller retains the object and the game ends.[4] If the highest signal $b \in [\beta(x_{J-1}), \beta(x_J))$, then the conditional density for the highest buyer's valuation is:

$$h(v|b) = \begin{cases} \frac{f(v)}{F(x_J)-F(x_{J-1})}, & x_{J-1} \leq v \leq x_J \\ 0, & else \end{cases} \tag{5}$$

The seller computes the price, p_J, according to (2). It is clear that $p_J \geq x_{J-1}$. Note also that $p_J < x_J$, $J \neq 0$. This is because $p_J \geq x_J$ gives zero revenue; however a price p_J, such that $0 < p_J < x_J$ makes a revenue of $p_J \times \frac{F(x_J)-F(p_J)}{F(x_J)-F(x_{J-1})}$, which is strictly greater than zero. We are now ready to characterize the partition equilibria of the PD game:

Theorem 1 *$(X^*(M^*), p^*)$ is a partition equilibrium in the PD game if and only if:*

(1) $M^* = 1$.

(2) *The seller's price p^* is a constant and satisfies $p^* = \frac{1-F(p^*)}{f(p^*)} = \frac{1}{\lambda(p^*)}$, where λ denotes the hazard rate.*

[4] Clearly we can extend the PD game by adding a sequential continuation game in which the seller then offers the object to the next randomly chosen buyer at the same price until one of the buyers accepts. If no buyer accepts, then the seller retains the object. However, the addition of such a continuation game leads to a potential complication. The acceptance by any subsequent buyer (or rejection by all buyers) conveys information to the seller that the valuation exceeds the price (or all valuations are less than the price). This may lead the revenue-maximizing seller to raise (lower) the price and we have in effect an alternating-offers bargaining game of incomplete information. There would also typically be a cost associated with the communication of each offer. We hence address in this paper only the case of a single offer.

Remark: Theorem 1 states that when buyers are certain that the seller will price discriminate, then their signals convey no information to the seller. The partition strategy is the coarsest one given by $X^*(M^*) = \{0,1\}$. The seller's posterior distribution over valuations, as defined by (5), is the same as the prior distribution F. Thus the seller responds in this "babbling" equilibrium by setting a price p^*. No buyer can do better by unilaterally deviating to another signal because all signals are equally uninformative.

Proof. We start by proving necessity. Consider a partition equilibrium $(X^*(M^*), p^*)$ where p^* is not necessarily constant, but depends on the signals. We first establish that the probability of winning is greater with a signal in an interval with a higher index J^*. Denote by W_{J^*} the probability of winning with a signal in the interval $[\beta(x^*_{J^*-1}), \beta(x^*_{J^*}))$. One of the events leading to winning with a signal in $[\beta(x^*_{J^*-1}), \beta(x^*_{J^*}))$ is the event that all other buyers have a valuation not greater than $x^*_{J^*-1}$; its probability is $G(x^*_{J^*-1})$. However, this is not the only event, as a buyer can win the item even if another buyer is in the same winning interval (in which case the winning buyer is chosen at random). Hence, $W_{J^*} > G(x^*_{J^*-1})$. Now consider $J^{*\prime} < J^*$. The event that all other buyers have a valuation not greater than $x^*_{J^{*\prime}} \leq x^*_{J^*-1}$ is a required condition for winning with a signal in $[\beta(x^*_{J^{*\prime}-1}), \beta(x^*_{J^{*\prime}}))$; however, it is strictly not sufficient, because in addition either other buyers have to signal in lower intervals, or a buyer has to share the probability of a win with others in the same interval. Hence $W_{J^{*\prime}} < G(x^*_{J^{*\prime}}) \leq G(x^*_{J^*-1}) < W_{J^*}$.

Consider the seller and let $p^*_{J^*}$ denote the price that is charged if the highest signal is from $[\beta(x^*_{J^*-1}), \beta(x^*_{J^*}))$. We claim that in equilibrium $p^*_{J^*} = p^*_{J^{*\prime}} = p^*$ for all $1 \leq J^{*\prime} < J^* \leq M^*$. Clearly $p^*_{J^{*\prime}} > p^*_{J^*}$ is not optimal. Therefore, let $p^*_{J^{*\prime}} < p^*_{J^*}$. Consider the expected payoff of a buyer with valuation $v \in [p^*_{J^*}, x^*_{J^*})$. A signal from the higher interval results in a payoff of $(v - p^*_{J^*}) \times W_{J^*}$, and a signal from the lower interval in a payoff of $(v - p^*_{J^{*\prime}}) \times W^*_{J^{*\prime}}$. Consider buyers with valuations $v \in [p^*_{J^*}, \min(p^*_{J^*} + \bar{v}, x^*_{J^*}))$, where

$$\bar{v} = \frac{p^*_{J^*} - p^*_{J^{*\prime}}}{W_{J^*} - W_{J^{*\prime}}} W_{J^{*\prime}}$$

Since $W_{J^*} > W_{J^{*\prime}}$, $\bar{v}$ is positive. For these buyers, the payoff from sending a signal from the lower interval is strictly greater than sending one from the higher interval. Hence these buyers have an incentive to deviate, contradicting the hypothesis that $(X^*(M^*), p^*)$ is an equilibrium. Finally, because $x_J > p_J \geq x_{J-1}$, $p^*_{J^*} = p^*_{J^{*\prime}} = p^*$ is only possible if $M^* = 1$, i.e. $X^*(M^*) = \{0,1\}$.

We now establish sufficiency. Given that $X^*(M^*) = \{0,1\}$, no information is conveyed to the seller. Thus the seller chooses the price p^* that maximizes $p(1 - F(p))$. A unilateral deviation by any buyer does not change payoffs because all signals convey the same coarsest information that valuations are distributed on $[0,1]$ according to F. ∎

It is clear that the equilibrium is in general inefficient. There are two sources of inefficiency. First, the seller may (randomly) allocate the object to a buyer who does not have the highest valuation. Second, since $p^* > 0$, with positive probability the seller may choose a buyer who will reject the object. Thus the seller will retain the object even though there are buyers with positive valuations.

4 The ϵ-Discriminating Seller

How should a price discriminating seller respond to a "babbling" equilibrium in which buyers convey no information in their signals? In this section, we demonstrate that the seller can induce the buyers to divulge more information if he chooses not to price discriminate with positive probability. We first make concrete what a non-discriminatory pricing strategy by the seller entails within the partition equilibrium framework:

Definition 3 *The seller's pricing strategy is non-discriminatory if the price charged is $\beta(x_{J-1})$ when the highest signaling interval observed is $[\beta(x_{J-1}), \beta(x_J))$ and $x_{J-1} > 0$. If the lower end of the highest interval is zero, then the seller retains the item.*

Remark: Recall that a price-discriminating seller would have charged a price according to (2) if $[\beta(x_{J-1}), \beta(x_J))$ was the highest signaling interval. In other words, the seller would have exploited the information in any signal from $[\beta(x_{J-1}), \beta(x_J))$ to deduce that the buyer's private valuation is at least x_{J-1} and responded with a revenue-maximizing price $p_J \geq x_{J-1} > \beta(x_{J-1})$. The non-discriminatory seller, on the other hand, is ignoring this information and restricting himself to a price equal to the lower end of the signaling interval. We will show that it is profitable for the seller to strategically ignore information in this manner.

We will accordingly consider a modified ϵ-PD game in which the seller employs a mixed strategy. The seller once again offers the object to those with signals in the highest interval. With probability ϵ, the price charged is discriminatory and maximizes the seller's revenue. With probability $1 - \epsilon$, the price is non-discriminatory and equal to the lower end of the signaling interval. Recall that in the PD game ($\epsilon = 1$), the buyers provide no information at all in their signals, i.e. there is a single partition in the signaling strategy. However, when $\epsilon \neq 1$, we show, under appropriate restrictions, that a partition signaling strategy with *two* partitions exists for all $\epsilon \in (0, 1)$. This suggests that the non-zero probability of choosing a non-discriminatory price motivates the buyers to provide more information to the seller.

With some abuse of notation, we denote the prices charged when the highest signals are in the intervals $[0, \beta(a))$ and $[\beta(a), \beta(1)]$ by $p_1(a)$ and $p_2(a)$ respectively.

Theorem 2 *A symmetric equilibrium signaling strategy with partition* $(0, \beta(a), \beta(1))$, *corresponding to the valuation partition* $(0, a, 1)$, *exists for all* $\epsilon \in (0, 1)$ *when* $\beta(1) \geq \frac{N-1}{N}$, $\lim_{x\to 0} \frac{\beta(x)}{x} < 1$ *and* $p_1(x) \leq \beta(x)$.

Remark: These sufficient conditions are satisfied when $F(v) = v$ and $\beta = \alpha$. With uniformly distributed values, $\beta(1) = \frac{N-1}{N} \geq \frac{N-1}{N}$ and $\lim_{x\to 0} \frac{\beta(x)}{x} = \frac{N-1}{N} < 1$. Further, $p_1(x) = \frac{x}{2} \leq \beta(x)$.

Proof. Consider the function $\frac{\psi(a)}{a} = \frac{\phi(b_2,a,a)-\phi(b_1,a,a)}{a}$, where $\phi(b, a, v)$ is the buyer's payoff for signal b, partition $(0, a, 1)$ and valuation a. It is the difference in payoff between signaling a higher valuation (one in $[a, 1]$) and a lower valuation (one in $[0, a)$), at the common point of these intervals, a, as a fraction of a. Noting that, when the seller price discriminates, the buyer with valuation a is either not able to afford the item, or makes a payoff of zero:

$$\psi(a) = (1-\epsilon)\left[1 - \frac{\beta(a)}{a}\right] W_2(a) - \left[1 - \frac{\epsilon p_1(a)}{a}\right] W_1(a)$$

where:

$$W_2(a) = \frac{\sum_{i=0}^{N-1} F^i(a)}{N}, \qquad W_1(a) = \frac{F^{N-1}(a)}{N}$$

The value of ψ is easily extended to make the function continuous over $[0, 1]$. In particular, by defining $\psi(0) = \lim_{a\to 0} \psi(a)$, which exists as both $p_1(a)$ and $\beta(a)$ are not greater than a, and both are continuous functions of a. Notice that $\psi(1) < 0$ because $p_1 < a$ and $\beta(1) \geq \frac{N-1}{N}$. Further, $\psi(0) > 0$. Hence, by the intermediate value theorem there exists $a^* \in (0, 1)$ such that $\psi(a^*) = 0$. Note that, because $\beta(a^*) \geq p_1(a^*)$, $1 - \frac{\beta(a^*)}{a^*} \leq 1 - \frac{\epsilon p_1(a)}{a^*}$, and, hence, $(1-\epsilon)W_2(a^*) \geq W_1(a^*)$. To see that the partition $(0, \beta(a^*), \beta(1))$ provides an equilibrium signaling strategy, consider a deviating buyer. The expected payoff of a buyer with valuation $a^* \pm \delta$ for $\delta > 0$ is:

$$\begin{aligned}\phi(b_2, a^*, a^* \pm \delta) &= (a^* \pm \delta - \beta(a^*))(1-\epsilon)W_2(a^*) \\ \phi(b_1, a^*, a^* \pm \delta) &= (a^* \pm \delta - \epsilon p_2(a^*))W_1(a^*)\end{aligned}$$

The difference

$$\phi(b_2, a^*, a^* \pm \delta) - \phi(b_1, a^*, a^* \pm \delta) = \pm\delta((1-\epsilon)W_2(a^*) - W_1(a^*))$$

is non-negative when $\delta > 0$, and non-positive otherwise. Hence a buyer has no incentive to deviate unilaterally. ■

In order to explore the implications of a two-partition equilibrium, we will now restrict ourselves to the case of uniformly distributed valuations.

Example 1: (*The Two Buyer Case*) Let $N = 2$ and $F(v) = v$. Suppose the signals are partitioned into intervals $[0, \frac{a}{2})$ and $\left[\frac{a}{2}, \frac{1}{2}\right]$, corresponding to valuation intervals $[0, a)$ and $[a, 1]$, and $\beta = \alpha$, i.e. we are using $\beta(v) = \frac{v}{2}$. Note that $p_2 = \max\left\{\frac{1}{2}, a\right\}$ and $p_1 = \frac{a}{2}$. The probability of winning with a signal in $[\frac{a}{2}, \frac{1}{2}]$ is $W_2 = a + \frac{1-a}{2} = \frac{a+1}{2}$. The probability of winning with a signal in $[0, \frac{a}{2})$ is $W_1 = \frac{a}{2}$. a^* is the solution to $\psi(a) = 0$ (Recall that, when the seller price discriminates, the buyer with valuation a is either not able to afford the item, or makes a payoff of zero):

$$\psi(a) = (1-\epsilon)\left[a - \frac{a}{2}\right]\left[\frac{a+1}{2}\right] - \left[a - \epsilon\frac{a}{2}\right]\frac{a}{2}$$

$\psi(a^*) = 0$ when $a^* = 1 - \epsilon$. Note also that $(1-\epsilon)W_2(a^*) > W_1(a^*)$. We now examine if a buyer has an incentive to unilaterally deviate from a partition strategy corresponding to a valuation partition of $[0, a^*)$ and $[a^*, 1]$. Note that valuations below $\frac{a^*}{2}$ cannot afford either price offered to buyers in the higher interval, but can afford one of the prices offered to buyers in the lower interval. Hence we consider only those with valuations larger than $\frac{a^*}{2}$. Consider a valuation $v = a^* \pm \delta$ for $\delta > 0$. Then

$$\phi(b_2, a^*, a^* \pm \delta) - \phi(b_1, a^*, a^* \pm \delta) = \pm\delta((1-\epsilon)W_2(a^*) - W_1(a^*))$$

which is positive when $v = a^* + \delta$ and negative otherwise. Hence a buyer has no incentive to deviate.

We can now observe some interesting comparative-statics. As ϵ increases, the size of the *upper* valuation interval increases (reducing to a "babbling" equilibrium for $\epsilon = 1$). *Therefore, as the probability that the seller will price discriminate increases, buyers at the higher end of the valuation interval are less inclined to distinguish themselves from those at the lower end and thus send signals that increasingly encompass the lower end of the signaling interval.* With a high probability of price discrimination, the surplus lost from losing the object is relatively small. On the other hand when ϵ decreases, the upper valuation interval shrinks. The buyers at the higher end of the valuation interval now become increasingly selective in terms of distinguishing themselves from those with lower valuations and are willing to signal their high valuations with high signals. Since the chances are high that the seller will not price discriminate, the opportunity cost of a lost item in terms of the surplus foregone is now relatively large. ■

Example 2: (*The N buyer Case*) A value of a^* can be determined for the case of N buyers, $N > 2$, in a manner similar to Example 1. This value is a solution to $\psi(x) = 0$.

$$\psi(a) = (1-\epsilon)\left[a - \frac{(N-1)}{N}a\right] W_2 - \left[a - \epsilon\frac{a}{2}\right] W_1$$

where:

$$W_2 = \sum_{i=0}^{N-1} {}^{N-1}C_i \frac{(1-a)^i a^i}{i+1} = \frac{\sum_{i=0}^{N-1} a^i}{N}$$

$$W_1 = \frac{a^{N-1}}{N}$$

Hence,

$$(1-\epsilon)\sum_{i=0}^{N-2} a^i - \left[(N-1) - \frac{(N-2)\epsilon}{2}\right] a^{N-1} = 0$$

which reduces to, for example:

$$(1-\epsilon) - a = 0$$

when $N = 2$, and

$$(1-\epsilon) + (1-\epsilon)a - \left(2 - \frac{\epsilon}{2}\right) a^2 = 0$$

when $N = 3$. We can now consider some comparative-statics with respect to the number of buyers, i.e. the impact of *competition* among the buyers on the amount of noise in the signaling strategies. Let $\epsilon = \frac{1}{2}$ and note that $a^* = 0.5$ for $N = 2$ and $a^* \approx 0.70$ (taking the positive root) for $N = 3$. Thus, for a given ϵ, an increase in the number of buyers shrinks the *upper* valuation interval. *Thus competition among the buyers leads those with higher valuations to become increasingly selective in differentiating themselves from those with lower valuations.* ■

We can now compare the seller's expected revenue from the 2-partition equilibrium in the ϵ-PD game to that in the babbling equilibrium of the PD game. In particular, we show that the seller gains when he sometimes strategically ignores the information contained in the signals.

Proposition 1 *Let F be a uniform distribution on the valuation interval and $N = 2$. For a 2-partition equilibrium, the optimal probability of price discrimination for the seller is $\epsilon^* = 0.4752$. The expected revenue of the seller in the 2-partition Bayesian-Nash equilibrium of the ϵ^*-PD game exceeds that in the PD game where the seller always price discriminates.*

Proof. Let $\epsilon = 1 - \delta$. We first show that expected revenue is an increasing function of δ in a 2-partition equilibrium for $0 < \delta < \frac{1}{2}$. If $\delta < \frac{1}{2}$, i.e. $\epsilon > \frac{1}{2}$, then, the revenue R is:

$$R = (1 - F(a^*)^2)[\epsilon p_2^* \frac{1 - F(p_2^*)}{1 - F(a^*)} + (1 - \epsilon)\frac{a^*}{2}] + F(a^*)^2 \epsilon p_1^* \times \left[\frac{F(a^*) - F(p^*)}{F(a^*)}\right]$$

Using: $a^* = 1 - \epsilon = \delta < \frac{1}{2}$, $p_2^* = \frac{1}{2}$, $p_1^* = \frac{a^*}{2} = \frac{\delta}{2}$

$$\begin{aligned} R &= (1 - \delta^2)\frac{1}{4} + (1 - \delta^2)\frac{\delta^2}{2} + (1 - \delta)\frac{\delta^3}{4} \\ &= \frac{1}{4} + \frac{\delta^2}{4}(1 + \delta - 3\delta^2) \end{aligned}$$

which can be shown to be increasing for $0 < \delta < \frac{1}{2}$. We next show that expected revenue is a decreasing function of δ for $\frac{1}{2} < \delta < 0$. If $\delta > \frac{1}{2}$, i.e. $\epsilon < \frac{1}{2}$, then:

$$R = \left(1 - F(a^*)^2\right)\left[\epsilon a^* + (1 - \epsilon)\frac{a^*}{2}\right] + F(a^*)^2 \epsilon p_1^* \times \left[\frac{F(a^*) - F(p^*)}{F(a^*)}\right]$$

Using: $a^* = 1 - \epsilon = \delta > \frac{1}{2}$, $p_2^* = a^*$, $p_1^* = \frac{a^*}{2} = \frac{\delta}{2}$

$$R = (1 - \delta^2)((1 - \delta)\delta + \frac{\delta^2}{2}) + \delta^2(1 - \delta)\frac{\delta}{4}$$

It can be shown that the derivative of R is zero at $\delta = 0.5248$, and negative thereafter. Therefore, the maximum expected revenue is achieved for $\delta^* = 0.5248$, and $\epsilon^* = 0.4752$, and its value is 0.2977. This value is greater than that of the babbling equilibrium ($\epsilon = 1$) as can be seen in Figure 1. ■

5 Conclusion

This paper was motivated by three main objectives. The first objective was to identify an interesting new class of first-degree price discrimination models that were distinguished by the fact that buyers send (possibly noisy) signals to the seller regarding their private valuation before a price is determined. The second objective was to demonstrate how the cheap talk framework could be adapted to characterize the equilibrium signaling strategy of the buyers and the pricing strategy of the seller. The third objective was to formally demonstrate that it was sometimes in the interest of the seller to strategically ignore price discrimination and instead adhere to a non-discriminatory pricing rule. Clearly there are a

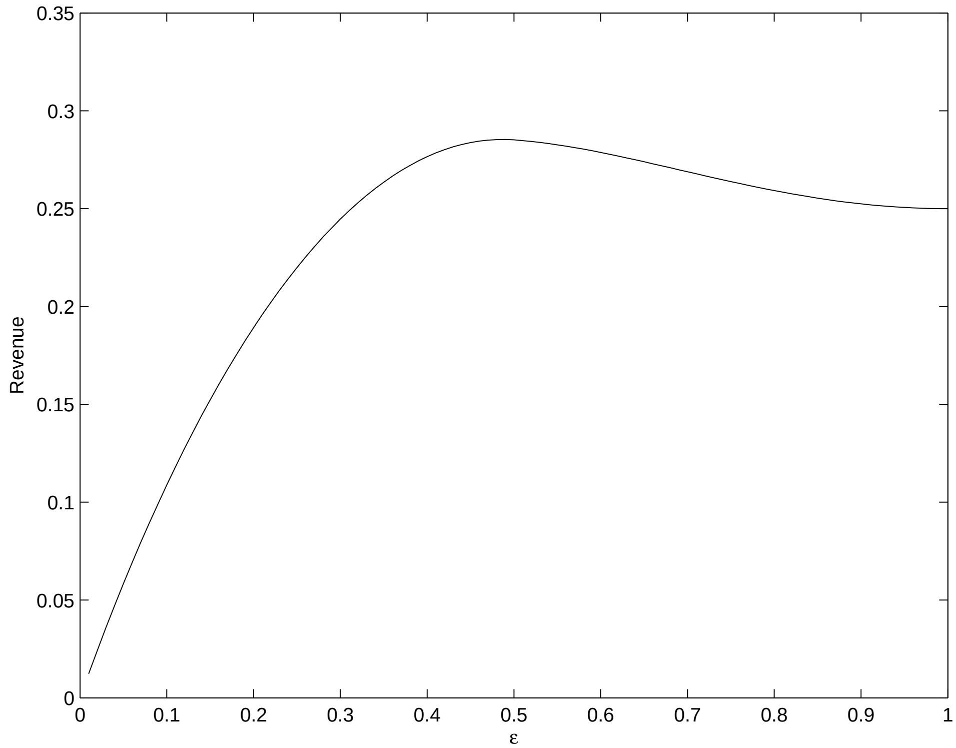

Figure 1: Revenue vs. ϵ; 2-buyer, 2-partition equilibria

number of interesting issues remaining to be addressed by building on the framework of this paper. For example, what is the finest partition equilibrium for any given $\epsilon \in (0, 1)$? How do buyer payoffs and seller revenue respond to an increase in the fineness of the partition equilibria for a given ϵ? What role will risk aversion or the heterogeneity of buyers play on the coarseness or fineness of the partition equilibrium? These remain productive avenues for future research.

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