Linear Combinations and Spans: Notes for CSci 124

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Definition A *linear combination* of n vectors $\mathbf{v}_i \in \mathbf{V}$ is a vector $\sum_{i=1}^{n} a_i \mathbf{v}_i$ for some $\{a_i\}_{i=1}^{n} \subset F$.

The values a_i above are often referred to as the *coefficients*.

Example 1:

$$2\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} - 7\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \\ 12 \end{bmatrix}$$

is a linear combination of

$$\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

A shorthand way of saying that a vector **v** is a linear combination of $\{\mathbf{v}_i\}_{i=1}^{n}$ is to say it lies in the *span* of these vectors. The *span* is the set of all possible linear combinations.

Definition The *span* of a finite set of vectors $\mathbf{B} = \{\mathbf{v}_i\}_{i=1}^{n} \subset \mathbf{V}$ is

$$Span(\mathbf{B}) = \{\sum_{i=1}^{n} a_i \mathbf{v}_i | a_i \in F\}$$

How does one determine if a given vector **v** belongs to the span of $\{\mathbf{v}_i\}_{i=1}^{n}$? Further, if it does, how does one determine the values a_i? Answer: solve a set of linear equations!

Determining if $\mathbf{v} \in Span(\{\mathbf{v}_i\}_{i=1}^{n})$

1. The question is, are there values a_1, a_2, ..., a_n such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_n\mathbf{v}_n = \mathbf{v}$$

2. Form a matrix **A** with $\{\mathbf{v}_i\}_{i=1}^{n}$ as its columns

$$\mathbf{A} = [\mathbf{v}_1 \mathbf{v}_2 .... \mathbf{v}_n]$$

3. Write a vector **a** containing the unknown coefficients a_i:

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ . \\ a_n \end{bmatrix}$$

4. Solve the set of linear equations $\mathbf{Aa} = \mathbf{v}$ using gaussian elimination

5. If there is exactly one solution, or many, $\mathbf{v} \in Span(\{\mathbf{v}_i\}_{i=1}^n)$, and the solutions provide the coefficients.

6. If there are no solutions, $\mathbf{v}$ is not in $Span(\{\mathbf{v}_i\}_{i=1}^n)$