Vector Spaces: Linear Independence Notes for CSci 124

Poorvi L. Vora

We reproduce Example 4 from the previous set of notes:

Example 4: Determine if

$$\begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

is a linear combination of

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{1}$$

If so, what are the coefficients? Again, in matrix form, the question is, find a solution to:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

if it exists. Gaussian elimination gives:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

where the final equation is of the form $0 = 0$. This means you have a consistent set of equations with several solutions. One solution is $a_3 = 1$, $a_2 = 0$, $a_1 = 3$. That is,

$$3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix} \tag{2}$$

Notice that three equations in three unknowns give several solutions. This happens because one of the equations is redundant in the presence of the other two: it provided no new information, which is why it disappeared on gaussian elimination. There were essentially two equations in three unknowns, and hence several solutions. This, in turn, happens because one of the vectors $\mathbf{v}_i$ is redundant. That is, any one of the three vectors $\{\mathbf{v}_i\}_{i=1}^{3}$ lies in the span of the other two, and hence a third vector does not provide an additional, independent direction. For example, of the three vectors in equation (1),

$$\frac{1}{2}\begin{bmatrix}1\\1\\1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}1\\-1\\1\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix}$$

That is, one can add some amounts of the vectors to cancel one another out, and get the zero vector. For example,

$$\frac{1}{2}\begin{bmatrix}1\\1\\1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}1\\-1\\1\end{bmatrix}-\begin{bmatrix}0\\1\\0\end{bmatrix}=\mathbf{0}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad (3)$$

We can add any amounts of the zero vector (equation (3)) to equation (2) to get several solutions to our problem. Suppose we add the zero vector n times, we get equation (2) + $n\times$ equation (3):

$$3\begin{bmatrix}1\\1\\1\end{bmatrix}+0\begin{bmatrix}1\\-1\\1\end{bmatrix}+1\begin{bmatrix}0\\1\\0\end{bmatrix}+n(\frac{1}{2}\begin{bmatrix}1\\1\\1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}1\\-1\\1\end{bmatrix}-\begin{bmatrix}0\\1\\0\end{bmatrix})=\begin{bmatrix}3\\4\\3\end{bmatrix}$$

or

$$(3+\frac{n}{2})\begin{bmatrix}1\\1\\1\end{bmatrix}-\frac{n}{2}\begin{bmatrix}1\\-1\\1\end{bmatrix}+(1-n)\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}3\\4\\3\end{bmatrix}$$

Hence, in general, $a_1 = 3+\frac{n}{2}$, $a_2 = -\frac{n}{2}$, $a_3 = 1-n$ for all integers n.

The above issue does not arise when the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are *linearly independent*. Loosely speaking, a set of vectors $\{\mathbf{v}_i\}_{i=1}^n$ is linearly independent if the only way of getting the zero vector by combining them is to take a zero amount of each vector.

1 Linear Independence

Definition A set of vectors $\{\mathbf{v}_i\}_{i=1}^n \subset \mathbf{V}$ is *linearly independent* if $\sum_{i=1}^n a_i\mathbf{v}_i = \mathbf{0} \Leftrightarrow a_i = 0\ for\ i = 1,2,...n$

Definition A set of vectors $\{\mathbf{v}_i\}_{i=1}^n \subset \mathbf{V}$ that is not linearly independent is *linearly dependent*.

Example 5: The vectors:

$$\mathbf{v}_1=\begin{bmatrix}2\\2\\3\end{bmatrix},\mathbf{v}_2=\begin{bmatrix}1\\3\\4\end{bmatrix},\mathbf{v}_3=\begin{bmatrix}3\\5\\7\end{bmatrix}$$

are linearly dependent because $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$. That is, because $\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$. Hence, $\sum_{i=1}^3 a_i\mathbf{v}_i = \mathbf{0}$ does not imply that $a_i = 0\ for\ i = 1,2,3$.

Example 6: The vectors we saw earlier,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

are also linearly dependent because

$$\frac{1}{2}\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$$

and $\sum_{i=1}^{3} a_i \mathbf{v}_i = \mathbf{0}$ does not imply that $a_i = 0 \; for \; i = 1, 2, 3$.

Example 7: On the other hand, the vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent because $\sum_{i=1}^{3} a_i \mathbf{v}_i = \mathbf{0}$ implies that $a_i = 0 \; for \; i = 1, 2, 3$.

In general, how would one determine if a set of vectors was linear independent? Again, linear equations!

2 Determining the Linear Independence of a Set of Vectors $\{\mathbf{v}_i\}_{i=1}^{n}$

1. The question is, are there values a_1, a_2, ..., a_n, not all zero, such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_n\mathbf{v}_n = \mathbf{0}$$

2. Form a matrix $\mathbf{A}$ with $\{\mathbf{v}_i\}_{i=1}^{n}$ as its columns

$$\mathbf{A} = [\mathbf{v}_1\mathbf{v}_2....\mathbf{v}_n]$$

3. Write a vector **a** containing the unknown coefficients a_i:

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ . \\ a_n \end{bmatrix}$$

4. Solve the set of linear equations $\mathbf{Aa} = \mathbf{0}$ using gaussian elimination. How many solutions does it have?

   (a) We know that there is one solution for these equations, it is $\mathbf{a} = 0$.

   (b) If there is exactly one solution, and no more, the vectors $\{\mathbf{v}_i\}_{i=1}^n$ are linearly independent.

   (c) If there are several solutions, the vectors $\{\mathbf{v}_i\}_{i=1}^n$ are not linearly independent

Example 8: Is the set of vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

linearly independent? In other words, the question is, are there values a_1, a_2, a_3 and a_4, not all zero, such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 = \mathbf{0}$$

or

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \mathbf{0}$$

Now, the above are four equations is four unknowns, for which we already know one solution: $a_1 = a_2 = a_3 = a_4 = 0$. So the set of vectors is linearly independent only if there are no more solutions. Gaussian elimination, as always.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \mathbf{0}$$

Swap for a non-zero pivot. Remember to swap rhs as well, though, in this case, because the rhs is zero, it doesn't matter.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \mathbf{0}$$

Because the final equation contains a single unknown, there is only one solution to this set of equations. We know that $a_1 = a_2 = a_3 = a_4 = 0$ is *a* solution, now we know it is the only solution. (You can check that the gaussian elimination gives you that solution: $-4a_4 = 0$, or $a_4 = 0$. Working backwards, you get that all four unknowns are zero.)

We could have determined the same by checking if the matrix corresponding to the equations was invertible. If it was, then there would be exactly one solution. We didn't invert the matrix because this works only when there are n equations in n unknowns, which is not always the case.

3 Spans and Linear Equations

Consider a set of n equations in m unknowns. We have observed examples of the following facts:

1. The set of equations $\mathbf{Ax} = \mathbf{b}$ has *a solution* (that is, at least one solution) if $\mathbf{b} \in Span\{\mathbf{A}_i\}_{i=1}^m$ where $\mathbf{A}_i$ represents the i^{th} column of $\mathbf{A}$. That is, the equations have a solution if **b** can be expressed as a linear combination of $\{\mathbf{A}_i\}_{i=1}^m$.
2. The converse is also true: the set if equations has no solution if **b** is not in $Span\{\mathbf{A}_i\}_{i=1}^m$.
3. If $\mathbf{b} \in Span\{\mathbf{A}_i\}_{i=1}^m$ (the set of equations has at least one solution), then:
   (a) The set of equations has more than one solution if $\{\mathbf{A}_i\}_{i=1}^m$ are not linearly independent.
   (b) The converse is also true. The set of equation has only one solution if $\{\mathbf{A}_i\}_{i=1}^m$ are linearly independent.

4 Bases of a Vector Space

Consider the vector space $\mathbb{R}^2$. Consider any vector in it. This vector can be written as a linear combination of

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x\mathbf{v}_1 + y\mathbf{v}_2$$

That is

$$\mathbb{R}^2 \subseteq Span(\{\mathbf{v}_1, \mathbf{v}_2\})$$

Further, there is no vector in $Span(\{\mathbf{v}_1, \mathbf{v}_2\})$ that is not in $\mathbb{R}^2$. Hence

$$\mathbb{R}^2 = Span(\{\mathbf{v}_1, \mathbf{v}_2\})$$

and we say that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a *basis* for $\mathbb{R}^2$. A *basis* of a vector space is a set of linearly independent vectors whose span is the vector space.

Definition A set of vectors $\mathbf{B} \subset \mathbf{V}$ is a basis for $\mathbf{V}$ if and only if $\mathbf{B}$ is linearly independent, and $Span(\mathbf{B}) = \mathbf{V}$.

In general, the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ . \\ . \\ . \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}, ...\mathbf{v}_i = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \\ . \\ . \\ . \\ 0 \end{bmatrix}, ...\mathbf{v}_n = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ . \\ . \\ . \\ 1 \end{bmatrix}$$

form a basis for $\mathbb{R}^n$, known as the *standard basis*.

Other bases exist, see Example 9 and Exercises 2 and 3 for examples.

5 Summary of Methodology to Determine if a set A is a Basis for a Vector Space V

1. Determine if the set **A** is linearly independent. If it is not, it is not a basis.

2. Determine if every vector in **V** can be expressed as a linear combination of the vectors in **A**. If so, you have shown that $\mathbf{V} \subseteq Span(\mathbf{A})$

3. Determine if every vector in $Span(\mathbf{A})$ belongs to **V**. If so, you have shown that $Span(\mathbf{A}) \subseteq \mathbf{V}$.

   In the problems we address, this will be easy. **V** will be one of $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$ or $\mathbb{R}^4$. As the vectors in **A** will be either one, two, three or four dimensional, it will be straightforward to say whether these vectors, and any linear combinations of them, belong to the $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$ or $\mathbb{R}^4$. See Example 9, and Exercises 2 and 3 for further clarification.

4. If 2 and 3 above hold, we have $\mathbf{V} = Span(\mathbf{A})$

5. If 1 and 4 above hold, **A** is a basis for **V**

Example 9:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

is a basis for $\mathbb{R}^4$.

We have already shown the above set is linearly independent (see Example 8). To show that it is a basis, we also need to show that $\mathbb{R}^4$ is its span. For this, we need to consider any vector in $\mathbb{R}^4$:

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

and show that it is a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$. That is, we need to show that there exist a_1, a_2, a_3, a_4 such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

or

$$a_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} + a_4 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

Again, the equations may be written as:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

which may be reduced:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} w \\ x-w \\ y-w \\ z-w \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} w \\ y-w \\ x-w \\ z-w \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} w \\ y-w \\ x-w \\ z-w-(y-w)=z-y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} w \\ y-w \\ x-w \\ z-y+x-w \end{bmatrix}$$

$a_4 = \frac{w-x+y-z}{4}$, $a_3 = \frac{w-x-y+z}{4}$, $a_2 = \frac{w+x-y-z}{4}$, $a_1 = \frac{w+x+y+z}{4}$

Thus, given any vector in $\mathbb{R}^4$, we have obtained its coefficients wrt $\{\mathbf{v}_i\}_{i=1}^4$, hence $\mathbb{R}^4 \subseteq Span(\{\mathbf{v}_i\}_{i=1}^4)$.

Further, because each of the $\mathbf{v}_i$ is a four dimensional column, linear combinations of the $\mathbf{v}_i$ cannot lie outside the four-dimensional space over the real numbers. Hence, $Span(\{\mathbf{v}_i\}_{i=1}^4) \subseteq \mathbb{R}^4$.

Hence $Span(\{\mathbf{v}_i\}_{i=1}^4) = \mathbb{R}^4$. Example 8 shows that $\{\mathbf{v}_i\}_{i=1}^4$ are linearly independent, hence $\{\mathbf{v}_i\}_{i=1}^4$ forms a basis for $\mathbb{R}^4$.

6 The Dimension of a Vector Space

Definition The number of vectors in a finite basis of a vector space is its *dimension*.

Fact: The number of vectors in a basis for $\mathbb{R}^n$ is n.

This gives us one additional fact connecting linear equations to vector spaces. Consider a set of n equations in n unknowns, represented by the matrix equation $\mathbf{A}\mathbf{x} = \mathbf{b}$. Notice that $\mathbf{A}$ is square. We now add to the observations of section 3 for the special case of the square matrix.

1. The equations have exactly one solution if $\mathbf{A}$ is invertible, or, equivalently, if all its columns are independent, or, equivalently, if the columns form a basis for $\mathbb{R}^n$.

7 Exercises

1. Which of the following sets of vectors is linearly independent? a.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

b.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

c.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

2. Show that

$$\mathbf{w}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

is a basis for $\mathbb{R}^3$.

3. Show that

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

is a basis for $\mathbb{R}^3$.

4. Express the vectors in (a)-(d) as linear combinations of the set of vectors:

$$\mathbf{w}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

a.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

b.

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

c.

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

d.

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$