Gaussian Elimination: Notes for CSci 124

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Gaussian Elimination is a simple algorithm for finding the solution to a set of linear equations. It is one of the most used of all algorithms for solving linear equations, because of its ease of implementation and robustness to error in the equations.

Let us look at one example where the set of equations has exactly one solution:

$$\begin{aligned} 3u + 5v + w &= 4 - - - - (1) \\ 6u + 11v + 4w &= 1 - - - - (2) \\ -3u + v - 2w &= 6 - - - - (3) \end{aligned}$$

We first use equation (1) to eliminate the variable u from equations (2) and (3). We do this by first identifying the pivot, which is the coefficient of u in equation (1). This should be non-zero. In this case, it is 3. We then subtract from equation (2), $\frac{6}{3}$ times equation (1), this guarantees that the coefficient of u will be zero in the "new" equation (2), which is:

$$(2) - \frac{6}{3} \times (1) : v + 2w = -7 - - - - (2)$$

Similarly, we subtract from equation (3), $\frac{-3}{3}$ times equation (1), this guarantees that the coefficient of u will be zero in the "new" equation (3), which is:

$$(3) - \frac{-3}{3} \times (1) : 6v - w = 10 - - - - (3)$$

Our new equations are now:

$$\begin{array}{ccccccccc} 3u & + & 5v & + & w & = & 4 & ---- & (1) \\ & & v & + & 2w & = & -7 & ---- & (2) \\ & & 6v & - & w & = & 10 & ---- & (3) \end{array}$$

We repeat this process to identify a pivot 1, and a "new" equation for equation (3) to get:

$$\begin{array}{ccccccccc} 3u & + & 5v & + & w & = & 4 & ---- & (1) \\ & & v & + & 2w & = & -7 & ---- & (2) \\ & & & - & 13w & = & 52 & ---- & (3) \end{array}$$

As the last equation contains only one non-zero term, we now "back-substitute" starting with the last equation, (3):

$$w = \frac{52}{-13} = -4$$

and substituting this value into all other equations:

$$\begin{array}{cccclcl} 3u & + & 5v & = & 4-(-4)=8 & ---- & (1) \\ & & v & = & -7-(2\times -4)=1 & ---- & (2) \end{array}$$

That is,

$$\begin{array}{cccclcl} 3u & + & 5v & = & 8 & ---- & (1) \\ & & v & = & 1 & ---- & (2) \end{array}$$

Then we go to the next equation up, that is, to (2):

$$v = \frac{1}{1} = 1$$

and substitute its value back into all previous equations:

$$3u = 8-(5\times 1) = 3 ----(1)$$

to get only one unsolved equation:

$$3u = 3 ----(1)$$

Going to this next,

$$u = \frac{3}{3} = 1$$

And the solution is $u=1, v=1, w=-4$.

1 More Formally

Suppose you have the following m equations in n unknowns, where $x_1, x_2, ...x_n$ denote the unknowns.

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n & = & b_1 ----(1) \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n & = & b_2 ----(2) \\ . & & . \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n & = & b_m ----(m) \end{array}$$

1.1 Algorithm Gaussian Elimination

We first examine the algorithm where the equations are such that exactly one solution exists. In particular, this requires that $m = n$, though this is not all that is required.

for($i = 1$ to $n - 1$)

$pivot = a_{ii}$

/* Assuming exactly one solution exists, $pivot \neq 0$. Use it to eliminate the i^{th} column in all later equations. That is, eliminate $a_{k,i}$ for all other rows k greater than i. Do this by subtracting $\frac{a_{k,i}}{pivot}$ times the i^{th} row*/

for ($k = (i + 1)$ to n)

$factor = \frac{a_{k,i}}{pivot}$

$row_k = row_k - factor * row_k$

endfor

endfor

/* Back substitute */

for($i = n$ downto 1)

$x_i = \frac{b_i}{a_{ii}}$

for($k = i - 1$ downto 1) $b_k -= x_i * a_{k,i}$

endfor

1.2 Exercises

Solve the following sets of linear equations using gaussian elimination:

1.
$$\begin{array}{ccccccc} 2u & + & 6v & - & 4w & = & 18 \\ 3u & + & 10v & - & 9w & = & 27 \\ 5u & - & 10v & + & 10w & = & -10 \end{array}$$

2.
$$\begin{array}{ccccccc} 3u & + & 8v & + & 4w & = & 7 \\ 3u & + & 10v & - & 2w & = & -3 \\ 6u & - & 10v & + & 10w & = & -8 \end{array}$$

3.
$$\begin{array}{ccccccc} u & + & v & + & 2w & = & 2 \\ 5u & + & 10v & + & 9w & = & 4 \\ 2u & - & 8v & + & 3w & = & 13 \end{array}$$

1.3 When the Algorithm of Section 1.2 Fails

The algorithm of section 1.2 fails when the value of *pivot* is zero, and it is not possible to divide by it. In this case, the first step is to find an equation with a non-zero pivot for the variable, to swap the positions of the equations, and to proceed. Consider the following example:

$$\begin{array}{rcrcrcr} u & + & 2v & - & 4w & = & -2 \\ 3u & + & 6v & - & 9w & = & -3 \\ 5u & - & 10v & + & 10w & = & -20 \end{array}$$

This reduces to:

$$\begin{array}{rcrcrcr} u & + & 2v & - & 4w & = & -2 \\ & & & & 3w & = & 3 \\ & - & 20v & + & 30w & = & -10 \end{array}$$

The second pivot is zero. Swap the second and third equations to get:

$$\begin{array}{rcrcrcr} u & + & 2v & - & 4w & = & -2 \\ & - & 20v & + & 30w & = & -10 \\ & & & & 3w & = & 3 \end{array}$$

and a pivot of -20. However, as the third equation does not have a term with v, this is the final form (note that, if there were four or more equations, the others would have terms with v). The solution is easily seen to be $u = -2$, $v = 2$ and $w = 1$.

On the other hand, if there is no remaining equation with a non-zero pivot for the variable, then the set of equations is *singular*. It either has no solutions, or an infinite number of solutions. In this case, proceed to the next variable, and proceed with elimination. At the very end, you will be left with at least two equations each consisting of the same, one, unknown. Reduce the equations as you normally would. If all equations that do not contain unknowns are of the form $0 = 0$, you have a set of consistent equations with an infinite number of solutions. See the following example:

$$\begin{array}{rcrcrcr} u & + & 2v & - & 4w & = & -2 \\ 3u & + & 6v & - & 9w & = & -3 \\ 5u & + & 10v & + & 10w & = & 20 \end{array}$$

This reduces to:

$$\begin{array}{rcrcrcr} u & + & 2v & - & 4w & = & -2 \\ & & & & 3w & = & 3 \\ & & & & 30w & = & 30 \end{array}$$

Notice that there is no other equation with a non-zero coefficient for v, hence the set of equations is singular. Proceed as you would, looking now for a pivot for the next variable, w:

$$\begin{array}{rcrcrcr} u & + & 2v & - & 4w & = & -2 \\ & & & & 3w & = & 3 \\ & & & & 0 & = & 0 \end{array}$$

and the set of equations is consistent, with and infinite number of solutions: $w = 1$, and $u + 2v = 2$.

If the previous set of equations had been slightly different, however, say, :

$$\begin{array}{rcrcrcr} u & + & 2v & - & 4w & = & -2 \\ 3u & + & 6v & - & 9w & = & -3 \\ 5u & + & 10v & + & 10w & = & 25 \end{array}$$

It would reduce to:

$$\begin{array}{rcrcrcl} u & + & 2v & - & 4w & = & -2 \\ & & & & 3w & = & 3 \\ & & & & 30w & = & 35 \end{array}$$

and

$$\begin{array}{rcrcrcl} u & + & 2v & - & 4w & = & -2 \\ & & & & 3w & = & 3 \\ & & & & 0 & = & 5 \end{array}$$

If there is any equation of the form $0 = c \neq 0$, the set of equations is inconsistent and has no solutions. This set of equations is inconsistent and has no solutions.

1.4 Exercises

4.

$$\begin{array}{rcrcrcl} 2u & + & 4v & + & 2w & = & 8 \\ 2u & + & 4v & + & 4w & = & 10 \\ u & - & v & + & 4w & = & 4 \end{array}$$

5.

$$\begin{array}{rcrcrcl} u & + & 3v & + & 2w & = & 0 \\ 2u & + & 6v & + & 4w & = & 0 \\ 2u & + & 4v & + & 4w & = & 10 \end{array}$$

6.

$$\begin{array}{rcrcrcl} u & + & 2v & + & w & = & 1 \\ u & + & v & - & w & = & -2 \\ u & + & 3v & - & 2w & = & 6 \end{array}$$