Complex Numbers: Notes for CSci 124

Poorvi L. Vora

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1 Motivation for the Definition of Complex Numbers

Suppose you wished to be able to count objects. You would start with the notion of one object, and then add another and so on. That is, you would want a set of numbers that contains the number 1 and is closed with respect to addition. This is the set of *natural numbers*

$$\mathcal{N} = \{1, 2, 3, ...\}$$

Notice that $\mathcal{N}$, is also closed under both addition and multiplication. That is,

$$x \in \mathcal{N} \; and \; y \in \mathcal{N} \Rightarrow xy \in \mathcal{N} \; and \; x + y \in \mathcal{N}$$

However, the converse is not true. That is, $x \in \mathcal{N} \; and \; y \in \mathcal{N}$ does not imply that $x - y \in \mathcal{N}$.

Example 1 Which of the following result in natural numbers? $5 - 3$; $10 - 20$; $7 - 1$; $29 - 30$?

$5 - 3 = 2$ is a natural number; $10 - 20 = -10$ is not; $7 - 1 = 6$ is; $29 - 30 = -1$ is not.

If, in addition to adding objects, you wish to also take away objects from an existing set, you need a set of numbers that includes $\mathcal{N}$ and is closed under subtraction. The smallest such set is the set of *integers*,

$$\mathcal{I} = \{..., -2, -1, 0, 1, 2, 3, ...\}$$

$\mathcal{I}$ is also said to be the *closure* of $\mathcal{N}$ under subtraction. The integers allow us to add and subtract objects, as well as to compare values – such as elevations and temperatures – with a predetermined zero value. However, just as $\mathcal{N}$ is not closed under subtraction, $\mathcal{I}$ is not closed under division: $x \in \mathcal{I} \; and \; y \in \mathcal{I}, y \neq 0$ does not imply that $\frac{x}{y} \in \mathcal{I}$.

Example 2: Which of the following result in integers? $3 - 2$; $4 - 5$; $\frac{5}{2}$; $\frac{25}{5}$; $\frac{-10}{-2}$?

$3 - 2 = 1$ is an integer; as are: $4 - 5 = -1$; $\frac{25}{5} = 5$; $\frac{-10}{-2} = 5$. $\frac{5}{2}$ is not an integer.

The closure of $\mathcal{I}$ under division (except by zero) is the set of *rational numbers*,

$$\mathcal{Q} = \{\frac{p}{q} \mid p, q, \in \mathcal{I}, q \neq 0\}$$

The rational numbers allow us to share objects into equal parts such that each part is not necessarily a whole. For example, 3 apples among 2 children.

1.1 Some properties of $\mathcal{Q}$

Now we see that $\mathcal{Q}$ satisfies the following properties with respect to both addition and multiplication. Let $\circ$ represent either addition or multiplication. Then:

1. *Closure under operation:* If $x, y \in \ \mathcal{Q}$, $x \circ y \ \in \ \mathcal{Q}$.
2. *Identity:* There is an element e, called the identity,such that $x \circ e = x \ \forall \ x$. For addition, $e = 0$, and for multiplication, $e = 1$.
3. *Inverse:* $\forall \ x$ except $x = 0$ when $\circ$ is multiplication, $\exists \ Inv(x)$ such that $x \circ Inv(x) = e$. For addition, $Inv(x) = -x$, and for multiplication, $Inv(x) = x^{-1}$.
4. *Closure under inverse:* $\mathcal{Q}$ is closed under the inverse operation when $\circ$ is addition, and $\mathcal{Q}\backslash\{0\}$ is closed under the inverse operation when $\circ$ is multiplication. (Here $\backslash$ denotes set subtraction.)

$\mathcal{Q}$ is, however, not closed with respect to square roots. That is, $x^2 = y \in \mathcal{Q}$ does not imply that $x \in \mathcal{Q}$. A physical implication of this is that all squares with rational values of area do not have rational values for side lengths.

For example, the square root of 2 cannot be written in the form $\frac{p}{q}$ for integers p and q. Yet, one can see that $\sqrt{2}$ should lie between 1 (the positive square root of 1) and 2 (the positive square root of 4), and it physically represents the side of a square of area 2.

Example 3: Which of the following equations have at least one rational solution?
$x^2 + 1 = 0$; $x^2 - 4 = 0$, $x^2 - \frac{9}{16} = 0$, $x^2 - \frac{2}{3} = 0$.

$x^2 - 4 = 0$ has rational solutions $x = 2$ and $x = -2$; $x^2 - \frac{9}{16} = 0$ has rational solutions $x = \frac{3}{4}$ and $x = \frac{-3}{4}$. The other two equations do not have any rational solutions.

The set of numbers represented by the number line, the real numbers $\mathcal{R}$, contains the square roots of all the natural (and positive rational) numbers. Thus, in particular, it contains all solutions to equations of the form $x^2 - a = 0$ where $a \in \mathcal{Q}$ and $a > 0$. $\mathcal{R}$ does not, however, contain any solutions to the equation $x^3 - 2 = 0$, or to $x^2 + 1 = 0$. While the motivation for studying the roots of such equations is not as obvious as the motivation for studying, say, the rational numbers, we will see that these roots are, indeed, important, when we study the mathematics required for processing audio and video.

Example 4: Which of the following equations have at least one real solution?
$x^2 + 1 = 0$; $x^2 - 4 = 0$, $x^2 - \frac{9}{16} = 0$, $x^2 - \frac{2}{3} = 0$.

All except $x^2 + 1 = 0$.

1.2 Complex Numbers

It has been shown that the set of complex numbers,

$$\mathcal{C} = \{x + iy \mid x, y, \in \mathcal{R},\ and\, i^2 + 1 = 0\}$$

contains the roots of all polynomials with real or complex coefficients. x is the *real part* of complex number $z = x + iy$, and y the *imaginary part*. They are denoted $Re(z)$ and $Im(z)$ respectively. The number iy is *imaginary*, while x is real. When $y = 0$, the complex number is a real number, that is, $\mathcal{R} \subset \mathcal{C}$. $\mathcal{C}$ is represented by the plane, and the complex number $x + iy$ corresponds to the point (x, y) in this plane (see Figure 1). In this plane, the number line, a common representation of $\mathcal{R}$, corresponds to the x axis.

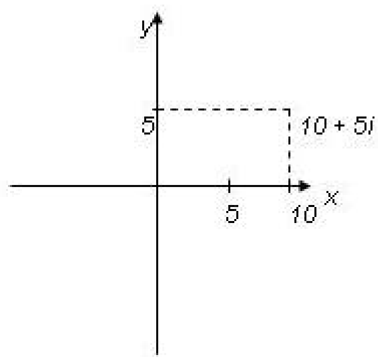

Figure 1: The complex number 10 + 5i on the complex plane

2 Operations on Complex Numbers

All operations on complex numbers are as those on real numbers, with the added condition that $i^2 = -1$.

1. *Equality:* $(x_1 + iy_1) = (x_2 + iy_2) \Rightarrow x_1 = x_2 \; and \; y_1 = y_2$
2. *Addition:* $(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$
3. *Subtraction:* $(x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$
4. *Multiplication:* $(x_1 + iy_1) \times (x_2 + iy_2) = x_1x_2 + iy_1x_2 + ix_1y_2 + i^2y_1y_2 = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)$

Division is slightly more complicated. A small trick helps. Notice that $(x+iy)\times(x-iy) = x^2+y^2$ is real. In fact, it is the *magnitude* of $x + iy$, and $x - iy$ is the *complex conjugate* of $x + iy$. Now,

$$\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1) \times (x_2 - iy_2)}{(x_2 + iy_2) \times (x_2 - iy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}$$

The *multiplicative inverse* may be similarly determined. Do in class.

Example 5:

A. Determine the following

1. $(2 + 3i) + (5 - 4i)$
2. $(3 + i) - (2 - i)$
3. $(5 - 6i) \times (3 - 2i)$
4. $(1 + 5i)^{-1}$
5. i^{-1}
6. $\frac{4+i}{3-i}$

B. Determine the roots of the quadratic equation $x^2 + 2x + 2 = 0$.

Answers:

A.

1. $(2 + 3i) + (5 - 4i) = (2 + 5) + (3 - 4)i = 7 - i$
2. $(3 + i) - (2 - i) = (3 - 2) + i(1 - -1) = 1 + 2i$
3. $(5 - 6i) \times (3 - 2i) = (5 \times 3 - (-6) \times (-2)) + ((-6) \times 3 + 5 \times (-2)) = 3 - 28i$
4. $(1 + 5i)^{-1} = \frac{1-5i}{5^2+1^2} = \frac{1-5i}{26}$
5. $i^{-1} = \frac{0-i}{0^2+1^2} = -i$

6. $\frac{4+i}{3-i} = \frac{4\times3-\{1\times(-(-1))\}+i(\{4\times(-(-1))\}+1\times3)}{3^2+1^2} = \frac{11+7i}{10}$

B. $x^2+2x+3=0 \Rightarrow x = \frac{-2\pm\sqrt{2^2-4\times1\times3}}{2\times1} = -1\pm\sqrt{2}i.$

Inverses

The multiplicative inverse of a complex number $x+iy$ is defined in a straightforward manner as that complex number, $x'+iy'$, which, when multiplied by $x+iy$ gives the multiplicative identity. That is:

$$(x+iy)\times(x'+iy') = 1$$

or

$$x'+iy' = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

3 Polar Representation

Some operations are not as easy to do in the Cartesian or Argand representation of complex numbers (i.e., using their x and y coordinates). It is easier, instead, to represent the complex number in terms of its *magnitude* and *phase*, *argument* or *angle* as shown in the diagram. The magnitude is the distance of the point(x,y) from the origin, and the angle is the angle made by the line joining the origin to the point with the x-axis in the anti-clockwise direction. If r is the magnitude and θ the angle, then:

$$\begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= tan^{-1}(\tfrac{y}{x}) \\ x &= rCos\theta \\ y &= rSin\theta \end{aligned}$$

The complex number is represented $r\angle\theta$ or $re^{i\theta}$. See Figure 2

Example 6:
a. Find the polar representation of the complex number $1+i$.
b. Find the cartesian representation of the complex number whose magnitude is 2, and angle 30°.
c. Find the polar representations of $1+i$, $1-i$, $-1+i$, $-1-i$.
d. Find the polar representations of $\sqrt{3}+i$, $\sqrt{3}-i$, $-\sqrt{3}+i$, $-\sqrt{3}-i$.

Answers:
a. $r = \sqrt{1^2+1^2} = \sqrt{2}$
$\theta = tan^{-1}(\frac{1}{1}) = tan^{-1}(1) = 45° = \frac{\pi}{4}$ radians. Hence, $1+i = \sqrt{2}\angle 45°$ or $\sqrt{2}\angle\frac{\pi}{4}^c$. Also denoted $\sqrt{2}e^{i\frac{\pi}{4}}$.
b. $2e^{i\frac{\pi}{6}} = 2\angle\frac{\pi}{6}^c = 2Cos(\frac{\pi}{6}) + i2Sin(\frac{\pi}{6}) = \sqrt{3}+i$
c. $\sqrt{2}e^{i\frac{\pi}{4}}$, $\sqrt{2}e^{\frac{-i\pi}{4}}$, $\sqrt{2}e^{i\frac{3\pi}{4}}$, $\sqrt{2}e^{i\frac{5\pi}{4}}$.
d. $2e^{i\frac{\pi}{6}}$, $2e^{i\frac{-\pi}{6}}$, $2e^{i\frac{5\pi}{6}}$, $2e^{i\frac{7\pi}{6}}$.

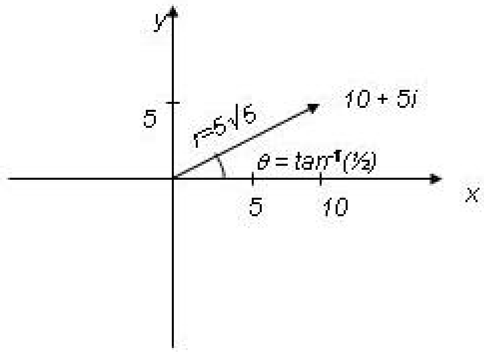

Figure 2: The complex number 10 + 5i in polar form

3.1 Multiplication

Recall that

$$(x_1 + iy_1) \times (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)$$

If $x_1 + iy_1 = r_1e^{i\theta_1}$, and $x_2 + iy_2 = r_2e^{i\theta_2}$, then

$$\begin{aligned} r_1e^{i\theta_1} \times r_2e^{i\theta_2} &= (x_1 + iy_1) \times (x_2 + iy_2) \\ &= r_1r_2(Cos\theta_1Cos\theta_2 - Sin\theta_1Sin\theta_2) + ir_1r_2(Sin\theta_1Cos\theta_2 + Cos\theta_1Sin\theta_2) \\ &= r_1r_2Cos(\theta_1 + \theta_2) + ir_1r_2Sin(\theta_1 + \theta_2) \\ &= r_1r_2e^{i(\theta_1+\theta_2)} \end{aligned}$$

That is, the product of two complex numbers results in a complex number with magnitude the product of the two magnitudes, and angle the sum of the two angles.

Example 7: Using the results of Example 6, find the polar representation of the product of the two numbers $1 + i$ and $\sqrt{3} + 1$. What is the cartesian representation?

Answer: From Example 6, we know that $1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$, and $\sqrt{3} + i = 2e^{i\frac{\pi}{6}}$. Hence

$$(1 + i) \times (\sqrt{3} + i) = 2\sqrt{2}e^{i\frac{5\pi}{12}} = (\sqrt{3} - 1) + i(\sqrt{3} + 1)$$

3.2 Complex Conjugation and Inverse

Note that the inverse of a complex number $re^{i\theta}$ is $\frac{1}{r}e^{-i\theta}$. This is seen easily. Recall that:

$$(x + iy)^{-1} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2} = \frac{x - iy}{r^2}$$

If the magnitude of $x+iy$ is r and its angle θ, the magnitude of $\frac{x-iy}{r^2}$ is $\frac{r}{r^2}=\frac{1}{r}$, and its angle is $-\theta$. Similarly, show that the conjugate of $re^{i\theta}$ is $re^{-i\theta}$.

Example 8: What is the polar representation of $2\sqrt{3}+2$? Hence, what is the polar representation of its inverse? Its complex conjugate? What are their cartesian representations?

Answer: $r=\sqrt{2^2\times 3+4}=\sqrt{16}=4$. $tan\theta=\frac{1}{\sqrt{3}}$ and point in the first quadrant, hence $\theta=\frac{\pi}{6}$. The polar representation of $2\sqrt{3}+2$ is $4e^{i\frac{\pi}{6}}$. The polar representation of its inverse is $\frac{1}{4}e^{\frac{-i\pi}{6}}$, and that of its conjugate: $4e^{\frac{-i\pi}{6}}$. Their cartesian representations are: $\frac{\sqrt{3}}{8}-\frac{1}{8}$ and $2\sqrt{3}+2$ respectively.