Linear Combinations and Spans: Notes for CSci 124/224

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Definition A *linear combination* of n vectors $\mathbf{v}_i \in \mathbf{V}$ is a vector $\sum_{i=1}^{n} a_i \mathbf{v}_i$ for some $\{a_i\}_{i=1}^{n} \subset F$.

The values a_i above are often referred to as the *coefficients*.

Example 1:

$$2 \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} - 7 \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \\ 12 \end{bmatrix}$$

is a linear combination of

$$\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

A shorthand way of saying that a vector $\mathbf{v}$ is a linear combination of $\{\mathbf{v}_i\}_{i=1}^{n}$ is to say it lies in the *span* of these vectors. The *span* is the set of all possible linear combinations.

Definition The *span* of a finite set of vectors $\mathbf{B} = \{\mathbf{v}_i\}_{i=1}^{n} \subset \mathbf{V}$ is

$$Span(\mathbf{B}) = \{\sum_{i=1}^{n} a_i \mathbf{v}_i | a_i \in F\}$$

How does one determine if a given vector $\mathbf{v}$ belongs to the span of $\{\mathbf{v}_i\}_{i=1}^{n}$? Further, if it does, how does one determine the values a_i? Answer: solve a set of linear equations!

1 Determining if $\mathbf{v} \in Span(\{\mathbf{v}_i\}_{i=1}^{n})$

1. The question is, are there values a_1, a_2, ..., a_n such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_n\mathbf{v}_n = \mathbf{v}$$

2. Form a matrix $\mathbf{A}$ with $\{\mathbf{v}_i\}_{i=1}^{n}$ as its columns

$$\mathbf{A} = [\mathbf{v}_1\mathbf{v}_2....\mathbf{v}_n]$$

3. Write a vector **a** containing the unknown coefficients a_i:

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ . \\ a_n \end{bmatrix}$$

4. Solve the set of linear equations $\mathbf{Aa} = \mathbf{v}$ using gaussian elimination

5. If there is exactly one solution, or many, $\mathbf{v} \in Span(\{\mathbf{v}_i\}_{i=1}^n)$, and the solutions provide the coefficients.

6. If there are no solutions, $\mathbf{v}$ is not in $Span(\{\mathbf{v}_i\}_{i=1}^n)$

2 Examples

Example 2: Determine if

$$\mathbf{v} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

belongs to $Span(\{\mathbf{v}_i\}_{i=1}^3)$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

We would proceed as follows. The question being asked is: are there values a_1, a_2, and a_3 such that

$$a_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

That is, does the set of equations:

$$\begin{array}{ccccccc} a_1 & + & a_2 & + & a_3 & = & 2 \\ a_1 & - & a_2 & + & a_3 & = & -2 \\ a_1 & + & a_2 & - & a_3 & = & 3 \end{array}$$

have a solution?

We can solve this using gaussian elimination right away:

$$\begin{array}{ccccccc} a_1 & + & a_2 & + & a_3 & = & 2 \\ & - & 2a_2 & & & = & -4 \\ & & & - & 2a_3 & = & 1 \end{array}$$

which gives us: $a_2 = 2$, $a_3 = -\frac{1}{2}$, and $a_1 = \frac{1}{2}$. Which means a solution does exist, and the given vector **v** may be expressed as a linear combination of the vectors $\{\mathbf{v}_i\}_{i=1}^3$.

We could also have written the above equation as a matrix equation. Recall, we did this when we were solving linear equations. The matrix is constructed with the vectors $\mathbf{v}_i$ as columns ($\mathbf{v}_1$ is the first column of the matrix, $\mathbf{v}_2$ is the second, and so on):

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

The above set of equations can be solved by trying to invert the matrix if it is square, or, as above, by using gaussian elimination, which is usually the quickest and cleanest approach.

In the above example, $\mathbf{v} \in Span(\mathbf{B})$. What happens when **v** *does not* belong to $Span(\{\mathbf{v}_i\}_{i=1}^n)$? Or when it does, but there are many possible values of $a_1, a_2, ...a_n$? Answer: the same thing that happens when a set of equations has no solutions, or has many solutions, respectively.

Example 3: Determine if

$$\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

belongs to $Span(\{\mathbf{v}_i\}_{i=1}^3)$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

If so, what are the coefficients? In matrix form, the question is, find a solution to:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

if it exists. If you proceed with gaussian elimination, you will be fine. If you try to invert the matrix, you won't be successful, because its determinant is zero. So always first attempt gaussian elimination. Don't forget to perform the same operations on the right-hand side.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$$

This gives you a final equation of the form $0 = 1$. This is an inconsistent set of equations, which means that **v** is not in $Span\{\mathbf{v}_i\}_{i=1}^3$.

Example 4: Determine if

$$\begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

is a linear combination of

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{1}$$

If so, what are the coefficients? Again, in matrix form, the question is, find a solution to:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

if it exists. Gaussian elimination gives:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

where the final equation is of the form $0 = 0$. This means you have a consistent set of equations with several solutions. One solution is $a_3 = 1$, $a_2 = 0$, $a_1 = 3$. That is,

$$3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix} \tag{2}$$