CSCI 124/224 - Discrete Structures II - Fall 2007
George Washington University

Homework 5 Solutions: 30 points, 6%

There will be no partial credit.

Solution Key (detailed explanations later)

1. (2 points) 30

2. (4 points: 1 per correct match)
   (i) d
   (ii) a
   (iii) c
   (iv) b

3. (5 points: 1 per correct match)
   (i) e
   (ii) b
   (iii) c
   (iv) a
   (v) d

4. (12 points: +1 per correct answer, -1 per incorrect answer)
   (a) False
   (b) True
   (c) True
   (d) True
   (e) False
   (f) False
   (g) False
   (h) True
   (i) False
   (j) True
   (k) False
   (l) True.

5. (3 points) 3 or 3 mod 5

6. (4 points, +1 for each correct answer). Yes. $a_1 = 1$, $a_2 = -1$, $a_3 = 1$

7. Both True and False get equal credit: +1

Detailed Solutions

1. (2 points: no partial credit) Find the smallest positive value of x that satisfies the following equations:

$$x = 2 \ mod \ 4$$
$$x = 3 \ mod \ 9$$

Solution: Use the Chinese Remainder theorem to get:

$$x = \sum_{i=1}^{2} y_i M_i a_i \ mod \ \Pi_{i=1}^{2} m_i = y_1 \times 9 \times 2 + y_2 \times 4 \times 3 \ mod \ 36 = 18y_1 + 12y_2 \ mod \ 36$$

$$y_1 = M_1^{-1} \ mod \ m_1 = 9^{-1} \ mod \ 4 = 1$$
$$y_2 = M_2^{-1} \ mod \ m_2 = 4^{-1} \ mod \ 9 = 7$$

$$x = 18 + 84 \ mod \ 36 = 102 \ mod \ 36 = 30$$

2. (4 points: 1 per correct match) Match the following. Note that some phrases may match several, however, you will eliminate all possibilities except one match by ensuring that all phrases have a match.

i. A vector in the span of $(1, 1, 1)$ and $(1, -1, 1)$

ii. A vector in the span of $(1, 1, 1)$

iii. $3 \times (1, 1, 1) + 4 \times (1, -1, 1)$

iv. A vector not in the span of $(1, 1, 1)$ and $(1, -1, 1)$

(a) (2, 2, 2)

(b) (1, 0, 0)

(c) (7, -1, 7)

(d) (2, 0, 2)

Solution:

i. A vector in the span of $(1, 1, 1)$ and $(1, -1, 1)$ is of the form $(x + y, x - y, x + y)$ for $x, y \in \mathbb{R}$. In particular, first and last coordinate should be identical. So (a), (c) and (d) are possibilities.

ii. A vector in the span of $(1, 1, 1)$ is of the form (α, α, α) for $\alpha \in \mathbb{R}$. The only possibility is (a).

iii. $3 \times (1,1,1) + 4 \times (1,-1,1) = (7,-1,7)$. The only possibility is (c). Hence the answer for (i) is (d).

iv. A vector not in the span of $(1,1,1)$ and $(1,-1,1)$. As observed in (i), (b) is the only possibility. It is the only one where the first and third coordinates are not identical.

3. (5 points: 1 per correct match) Match the following. Note that some phrases may match several, however, you will eliminate all possibilities except one match by ensuring that all phrases have a match.

i. v_1 and v_2 are linearly independent. The vector $\mathbf{0}$ is $\mathbf{0} = a_1 v_1 + a_2 v_2$. What is a possible value of (a_1, a_2)?

ii. $v_1 = (3,3)$ and $v_2 = (1,1)$. The vector $\mathbf{0}$ is $\mathbf{0} = a_1 v_1 + a_2 v_2$. What is a possible value of (a_1, a_2)?

iii. What is a possible vector orthogonal to $(1,-1)$?

iv. Let $v_1 = (1,0)$. If $\{v_1, v_2\}$ forms a basis for $\mathbb{R}^2$, what is a possible value for v_2?

v. $v_1 = (1,-1)$, and $v_2 = (2,-2)$. v_3 lies in $Span(\{v_1, v_2\})$. What is a possible value for v_3?

(a) (0, 3)
(b) (-2, 6)
(c) (5, 5)
(d) (5, -5)
(e) (0, 0)

Solution:

i. Because v_1 and v_2 are linearly independent, the only possibilities for (a_1, a_2) is $(0,0)$ (e)

ii. Solving for a_1 and a_2:

$$3a_1 + a_2 = 0$$

The only possibility is (b)

iii. Possible vector orthogonal to $(1,-1)$ is of the form (α, α) for $\alpha \in \mathbb{R}$. (c) and (e) are possible solutions, but (e) is taken. (c)

iv. Any vector linearly independent from v_1 is a possibility. All except (e) are possible. Hence (a) or (d)

v. Notice that a linear combination of v_1 and v_2 gives a vector of the form $(\alpha, -\alpha)$. Hence (d). Solution to (iv) is (a).

4. Say whether the following are True or False (12 points: +1 per correct answer, -1 per incorrect answer)

   (a) The vectors $v_1 = (1, 1)$ and $v_2 = (2, 2)$ are linearly independent

   False. One is a multiple of the other.

   (b) The vectors $v_1 = (1, 1)$ and $v_2 = (1, -1)$ are linearly independent

   True, as one is not a multiple of the other.

   (c) The vector $(a, 0, a)$ for any real value a is in the span of $(1, 1, 1)$ and $(1, -1, 1)$

   True. A vector in the span of $(1, 1, 1)$ and $(1, -1, 1)$ is of the form $(x + y, x - y, x + y)$. $x - y = 0 \Rightarrow x = y$, and $x + y = a \Rightarrow x = y = \frac{a}{2}$.

   (d) The vector $(0, a, 0)$ for any real value a is in the span of $(1, 1, 1)$ and $(1, -1, 1)$

   True. A vector in the span of $(1, 1, 1)$ and $(1, -1, 1)$ is of the form $(x + y, x - y, x + y)$. $x + y = 0 \Rightarrow x = -y$, and $x - y = a \Rightarrow x = -y = \frac{a}{2}$.

   (e) The vector $(a, a, 0)$ for any real value a is in the span of $(1, 1, 1)$ and $(1, -1, 1)$

   False. A vector in the span of $(1, 1, 1)$ and $(1, -1, 1)$ is of the form $(x + y, x - y, x + y)$. $x + y = 0$ and $x + y = a \Rightarrow a = 0$, but we wish any value of a.

   (f) Suppose v_1, v_2, $v_3 \in \mathbb{R}^3$, and v_3 lies in the same plane as v_1 and v_2. v_1, v_2 and v_3 are linearly independent.

   False, as v_3 can be expressed as a linear combination of v_1 and v_2, hence the vectors are not linearly independent.

   (g) Suppose $v_1 = v_2 + v_3$. v_1, v_2 and v_3 are linearly independent.

   False, as $a_1 = 1$, $a_2 = -1$, and $a_3 = -1$ gives $\sum a_i v_i$ is the zero vector.

   (h) Let $v_1 = (1, 1)$ and $v_2 = (2, 1)$. For any $v_3 = (x, y)$ for real x and y, there are unique coefficients a_1 and a_2 such that $v_3 = a_1 v_1 + a_2 v_2$.

   True.

$$\begin{aligned} a_1 + 2a_2 &= x \\ a_1 + a_2 &= y \end{aligned}$$

$$\begin{aligned} a_1 + 2a_2 &= x \\ a_2 &= x - y \end{aligned}$$

$$a_1 + 2x - 2y = x \Rightarrow a_1 = 2y - x$$

(i) Let $v_1 = (1, 1)$ and $v_2 = (2, 2)$. For any $v_3 = (x, y)$ for real x and y, there are unique coefficients a_1 and a_2 such that $v_3 = a_1v_1 + a_2v_2$.

False

$$\begin{aligned} a_1 + 2a_2 &= x \\ a_1 + 2a_2 = y \end{aligned}$$

Only possible if $x = y$.

(j) Let $v_1 = (1, 1)$ and $v_2 = (2, 2)$. There are coefficients a_1 and a_2 such that $a_1v_1 + a_2v_2 = (0, 0)$.

True. $a_1 = 0$ and $a_2 = 0$

(k) Suppose $v_3 = 2v_1 + v_2 = v_1 + 2v_2$. v_1 and v_2 are linearly independent.

False because $v_1 = v_2$. Hence $\exists a_1, a_2$, not all zero, such that $a_1v_1 + a_2v_2 = 0$. For example, $a_1 = 1$ and $a_2 = -1$.

(l) Two vectors in $\mathbb{R}^2$ that do not lie along the same line span $\mathbb{R}^2$.

True.

5. (3 points) f is a homomorphism from G_1 (the group of integers *mod* 4 with the operation $\diamond$, addition *mod* 4) to G_2 (the group of non-zero integers *mod* 5 with the operation $\circ$, multiplication *mod* 5). If $f(3) = 2$, what is $f(1)$?

Solution: $f(x + y \ mod\ 4) = f(x) \times f(y) \ mod\ 5$.

$$f(3) = f(1 + 2 \ mod\ 4) = f(1) \times f(2) \ mod\ 5$$

$$f(2) = f(1 + 1 \ mod\ 4) = f(1) \times f(1) \ mod\ 5$$

$$f(3) = f(1) \times f(1) \times f(1) \ mod\ 5 = f(1)^3 \ mod\ 5$$

Try all 4 possibilities for $f(1)$ and see which one satisfies above. $0^3 = 0$ *mod* 5; $1^3 = 1$ *mod* 5, $2^3 = 8$ *mod* $5 = 3$; $3^3 = 27$ *mod* $5 = 2$. Hence $f(1) = 3$.

6. (4 points) Is $v_4 = (0, 2, 1)$ in $Span(\{v_1 = (0, 1, 0), v_2 = (1, 1, 0), v_3 = (1, 2, 1)\})$? If so, what are the coefficients a_i, $i = 1, 2, 3$, in $v_4 = \sum a_i v_i$?

   Solve:

$$\begin{aligned} a_2 + a_3 &= 0 \\ a_1 + a_2 + 2a_3 &= 2 \\ a_3 &= 1 \end{aligned}$$

   $a_1 = 1$, $a_2 = -1$, $a_3 = 1$.