CSCI 124 - Discrete Structures II - Spring 2006
George Washington University

Extra Credit HW Solutions

1. (7 points) Construct a subset of the two-dimensional vector space over $\mathbb{R}$ that is closed under scalar multiplication, but not under vector addition. Justify your answer.

Solution: The union of the first and third quadrants. $\mathbf{A} = \{(x,y)|xy \geq 0\}$.

Consider $(x,y) \in V$, and scalar $c \in \mathbb{R}$. Then $xy \geq 0$. Also, $c(x,y) = (cx, cy)$ and $(cx)(cy) = c^2xy \geq 0$. Hence $c(x,y) \in \mathbf{A}$. Hence $\mathbf{A}$ is closed under scalar multiplication.

Consider $(1,0)$ and $(0,-1)$. Both belong to $\mathbf{A}$. But the sum, $(1,-1)$ is not in $\mathbf{A}$. Hence $\mathbf{A}$ is not closed wrt vector addition.

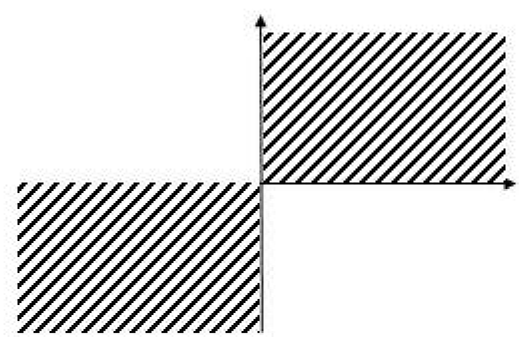

Figure 1: Quadrants I and III

Another possible answer is two distinct lines through the origin, or the two axes.

The union of the x and y axes in two-dimensional euclidean space over the field of real numbers, $\mathbb{R}$.

$$\mathbf{A} = \{(x,0)|x \in \mathbb{R}\} \cup \{(0,y)|y \in \mathbb{R}\} = \{(x,y)|xy = 0\}$$

Take any vector $b = (x,y)$ from $\mathbf{A}$, and any scalar c from $\mathbb{R}$. Then $c(x,y) = (cx, cy)$. As $c, x, y, \in \mathbb{R}$, $cx, cy \in \mathbb{R}$. Further, as $(x,y) \in \mathbf{A}$, either $xy = 0$. Hence $c^2xy = 0$. Hence $cb \in \mathbf{A}$, and $\mathbf{A}$ is closed wrt scalar multiplication.

Take any two non-zero vectors from $\mathbf{A}$ of the form $b' = (x,0)$ and $b = (0,y)$ $(x \neq 0; y \neq 0$. Their sum is (x,y), such that neither x nor y is zero. This is not in $\mathbf{A}$. Hence $\mathbf{A}$ is not closed wrt vector addition.

2. (7 points each) Which of the following subsets of $\mathbb{R}$ are subspaces (provide careful justification of your answer):

a. The plane of vectors $b = (b_1, b_2, b_3)$ with first component $b_1 = 0$.

Solution: The sum of any two vectors whose first component is zero also has first component zero. The sum of any two vectors whose components are real numbers also has real components. Hence this set is closed under vector addition. See Figure 3.

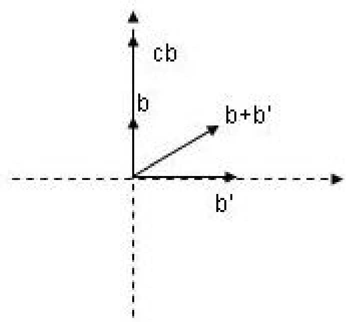

Figure 2: Problem 1 Second Solution: Dotted lines make up **A**

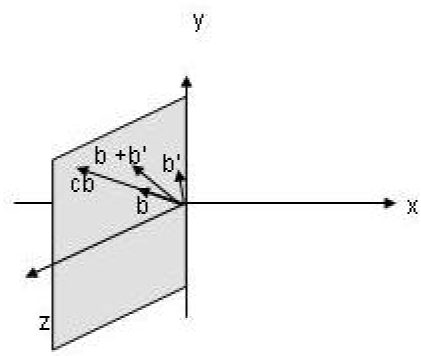

Figure 3: Problem 2 a

More formally, let $b, b' \in \mathbf{A}$, the set described above, such that $b = (b_1, b_2, b_3)$ and $b' = (b'_1, b'_2, b'_3)$. Then

$$b + b' = (b_1 + b'_1, b_2 + b'_2, b_3 + b'_3)$$

Further,

$$b_1 = b'_1 = 0 \Rightarrow b_1 + b'_1 = 0 \tag{1}$$

and

$$b_2, b'_2, b_3, b'_3, \ \in \ \mathbb{R} \Rightarrow b_2 + b'_2, b_3 + b'_3 \ \in \ \mathbb{R} \tag{2}$$

From equations (1) and (2), $b + b' \ \in \ \mathbf{A}$, which is hence closed under vector addition.

The scalar multiplication of a vector with first component zero also has first component zero. The multiplication of a real scalar with real components also results in real components. Hence the set is closed wrt scalar multiplication.

More formally, consider any vector $b = (b_1, b_2, b_3) \in \mathbf{A}$, and any $c \in \mathbb{R}$. Then

$$cb = (cb_1, cb_2, cb_3)$$

Further,

$$b_1 = 0 \Rightarrow cb_1 = 0 \tag{3}$$

and

$$c, b_2, b_3 \in \mathbb{R} \Rightarrow cb_2 \in \mathbb{R}, \ cb_3 \in \mathbb{R} \tag{4}$$

From equations (3) and (4), $cb \in \mathbf{A}$, which is hence closed under scalar multiplication.

Yes, $\mathbf{A}$ is a subspace of $\mathbb{R}^3$.

b. The plane of vectors $b = (b_1, b_2, b_3)$ with first component $b_1 = c \neq 0$.

Solution: The sum of any two vectors whose first component is the same non-zero value, c, has first component $2c \neq c$. Hence this set is not closed under vector addition. Hence it is not a subspace. See Figure 4.

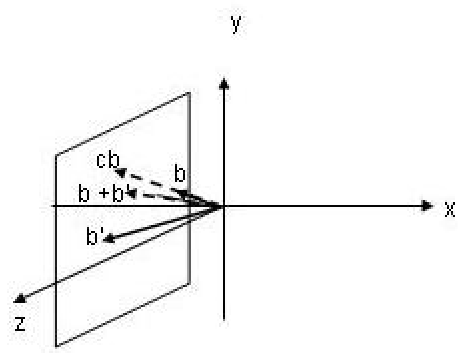

Figure 4: Problem 2 b: Notice that both the sum of the two vectors b and b' and the scalar multiplication of b' with c lie behind the plane and are shown with dotted lines.

More formally, let $b, b' \in \mathbf{A}$, the set described above, such that $b = (b_1, b_2, b_3)$ and $b' = (b'_1, b'_2, b'_3)$. Then

$$b + b' = (b_1 + b'_1, b_2 + b'_2, b_3 + b'_3)$$

Further,

$$b_1 = b'_1 = c \Rightarrow b_1 + b'_1 = 2c \neq c$$

Hence $b + b'$ is not in $\mathbf{A}$, which is not closed under vector addition.

No, $\mathbf{A}$ is not a subspace of $\mathbb{R}^3$.

c. The vectors $b = (b_1, b_2, b_3)$ with $b_1 \times b_2 = 0$ (that is, the product of b_1 and b_2 is zero).

Solution: Let $b, b' \in \mathbf{A}$, the set described above, such that $b = (b_1, b_2, b_3)$ and $b' = (b'_1, b'_2, b'_3)$. Then

$$b + b' = (b_1 + b'_1, b_2 + b'_2, b_3 + b'_3)$$

$$(b_1 + b'_1) \times (b_2 + b'_2) = b_1b_2 + b'_1b'_2 + b'_1b_2 + b_1b'_2 = b'_1b_2 + b_1b'_2$$

which need not be zero. See Figure 5.

For example, let $b = (0, 1, 1)$ and $b' = (1, 0, 1)$. Then $b, b' \in \mathbf{A}$, but $b + b' = (1, 1, 2)$ is not in $\mathbf{A}$. Hence $\mathbf{A}$ is not closed under vector addition.

No, $\mathbf{A}$ is not a subspace of $\mathbb{R}^3$.

d. The solitary vector $b = (0, 0, 0)$

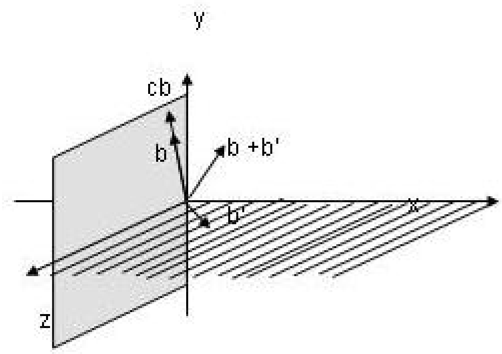

Figure 5: Problem 2 c: One shaded area corresponds to $b_1 = 0$ (the y-z plane), and the other to $b_2 = 0$ (the x-z plane).

Solution: Adding a zero vector to itself gives the zero vector. Multiplying the zero vector by any scalar gives the zero vector. Hence the set consisting of the zero vector is closed under vector addition and scalar multiplication.

Yes, the solitary vector $b = (0, 0, 0)$ is a subspace of $\mathbb{R}^3$.

e. All combinations of two given vectors, $x = (1, 1, 0)$ and $y = (2, 0, 1)$

Solution: The sum of two combinations of the above vectors will also be a combination of the vectors. Similarly, the scalar multiplication of a combination of the two vectors will also be a combination. See Figure 6.

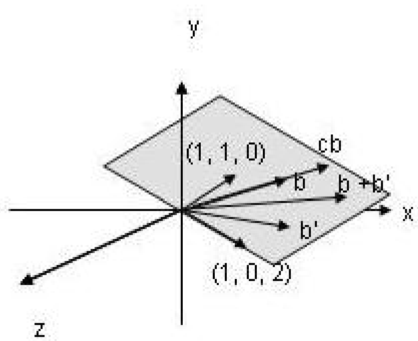

Figure 6: Problem 2 e: The subspace is the plane containing the two vectors

More formally, let $b, b' \in \mathbf{A}$, the set described above, such that $b = a_1 x + a_2 y$ and $b' = a'_1 x + a'_2 y$. Then,

$$b + b' = (a_1 + a'_1)x + (a_2 + a'_2)y$$

is a linear combination of x and y and belongs to $\mathbf{A}$. Hence $\mathbf{A}$ is closed under vector addition.

Further, let $c \in \mathbb{R}$. Then

$$cb = ca_1 x + ca_2 y$$

is also a linear combination of x and y and hence belongs to $\mathbf{A}$. Hence $\mathbf{A}$ is closed under scalar multiplication.

(Note that neither property above – of A being closed with respect to vector addition/scalar multiplication depends on what the vectors x and y are. They could have been any vectors.)

Yes, $\mathbf{A}$ is a subspace of $\mathbb{R}^3$.

f. The vectors $b = (b_1, b_2, b_3)$ that satisfy $b_3 - b_2 + 3b_1 = 0$.

Answer: Let $b, b' \in \mathbf{A}$, the set described above, such that $b = (b_1, b_2, b_3)$ and $b' = (b'_1, b'_2, b'_3)$. Then

$$b + b' = (b_1 + b'_1, b_2 + b'_2, b_3 + b'_3)$$

Further,

$$b_1, b'_1, b_2, b'_2, b_3, b'_3, \ \in \ \mathbb{R} \Rightarrow b_1 + b'_1, b_2 + b'_2, b_3 + b'_3 \ \in \ \mathbb{R} \tag{5}$$

$$(b_3 + b'_3) - (b_2 + b'_2) + 3(b_1 + b'_1) = (b_3 - b_2 + 3b_1) + (b'_3 - b'_2 + 3b'_1) = 0 + 0 = 0 \tag{6}$$

From equations (5) and (6), $b + b' \ \in \ \mathbf{A}$, which is hence closed under vector addition.

Further, if $c \in \mathbb{R}$,

$$cb = (cb_1, cb_2, cb_3)$$

And

$$c, b_1, b_2, b_3, \ \in \ \mathbb{R} \Rightarrow cb_1, cb_2, cb'_3 \ \in \ \mathbb{R} \tag{7}$$

$$(cb_3) - (cb_2) + 3(cb_1) = c(b_3 - b_2 + 3b_1) = c.0 = 0 \tag{8}$$

From equations (7) and (8), $cb \ \in \ \mathbf{A}$, which is hence closed under scalar multiplication.

Yes, $\mathbf{A}$ is a subspace of $\mathbb{R}^3$.

3A. Consider a square with vertices a, b, c, d. Consider the set of clockwise rotations of the square by multiples of 90°: $g_1 : 90°$, $g_2 : 180°$, $g_3 : 270°$, $g_4 : 360°$ (see Figure 7).

Let $G = \{g_1, g_2, g_3, g_4\}$. Define an operation on elements of G as follows: $g_i \diamond g_j$ is the rotation obtained by performing first g_i and then g_j. For example, $g_1 \diamond g_2 = g_3$, see Figure 8.

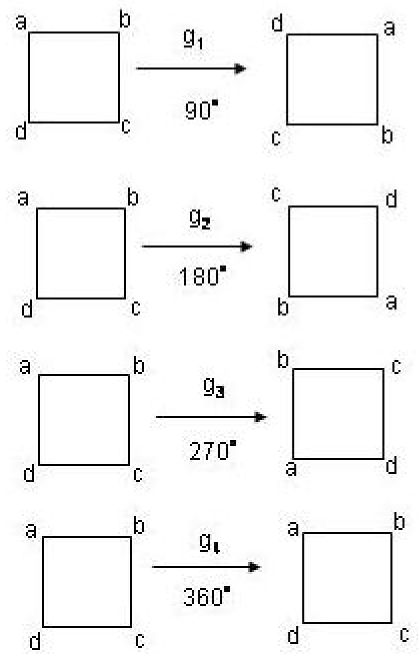

Figure 7: Rotations

It is known that G is a group under this operation. You do not need to prove this. Answer the following:

1. (5 points) What is the identity element of the group? Why?

   Solution: The identity is g_4, as, composed with any other operation g, the result is as though only g were applied.

2. (5 points) Which element g is such that $g \diamond g = e$ where e is the identity?

   Solution: g_2, as two rotations of 180° are equivalent to one of 360°, which is the identity.

3. (10 points) Recall that a subgroup of G is a subset of G that satisfies all the group conditions with respect to $\diamond$. The trivial subgroups of G are $\{e\}$ and G. Does G contain a non-trivial subgroup? If not, why? If it does, what is it?

   Solution: Yes, G does contain a non-trivial subgroup. It is the subgroup: $\{e = g_4, g_2\}$. g_2 composed with itself is e, and g_2 is also its own inverse.

B. Consider an isosceles triangle, and the group consisting of its rotations, clockwise, by multiples of 120°.

1. (10 points) Does this group contain a non-trivial subgroup? Why or why not?

   Solution: No, it doesn't. The clockwise rotation of 120° composed with itself gives the clockwise rotation of 240°, and vice versa. Hence any group containing one of these two must contain both. Hence any group that is not the identity alone, contains one of these and is hence the entire group.

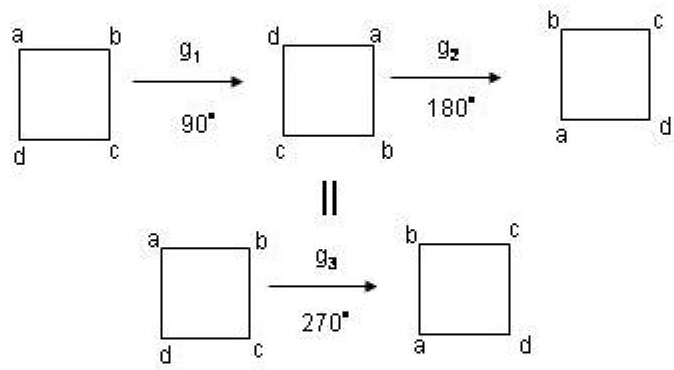

Figure 8: $g_1 \diamond g_2$

2. (5 points) Does this group contain an element g such that $g \diamond g = e$?

   Solution: The only such element is e itself.

3. (10 points) Consider a regular n-gone: an n-sided polygon with all sides equal. A rotation by $\frac{2\pi}{n}^c$ (note: this is not degrees as listed in the original homework) moves one vertex to the next one. Consider the group of rotations of all multiples of $\frac{2\pi}{n}^c$. For what values of n will this group contain an element g such that $g^2 = e$?

   Solution: The identity is a rotation of $2\pi^c$. If g is a rotation of $\frac{2i\pi}{n}^c$ for some integer i $0 < i \leq n$, then g^2 is a rotation of $\frac{4i\pi}{n}^c$. For this to be the identity, it should be a multiple of $2\pi^c$, that is, $\frac{4i\pi}{n} = 2k\pi$ for some integer k, or $\frac{2i}{n} = k$, or i is a multiple of $\frac{n}{2}$. As $i \leq n$, $i = \frac{n}{2}$ or $i = n$. The former is possible only when n is even, and the latter implies $g = e$. Thus this group contains such an element which is not the identity only if n is even.

4. Consider the group:

- $G = \mathbb{Z}_m$
- group operation $\diamond$ is addition *modulo* m

e denotes the identity in G.

Recall that a subgroup of G is a subset of G that satisfies all the group conditions with respect to $\diamond$. The trivial subgroups of G are $\{e\}$ and G.

You may not use Lagrange's theorem for any part of this question as it has not been covered in class.

a. (3 points) Provide an element $g \neq e$ of G such that $g \diamond g = e$ when $m = 6$.

Solution: $g = 3$, as $3 \diamond 3 = 3 + 3$ *mod* $6 = e$

b. (5 points) Show that G does not contain an element $g \neq e$ such that $g \diamond g = e$ when m is an odd number.

Solution: Proof by contradiction. Suppose G does contain an element $g \neq e$ such that $g \diamond g = e$ when m is an odd number. Then

$$\begin{aligned} g \diamond g &= e \\ \Rightarrow 2g &= 0 \ mod \ m \\ \Rightarrow 2g &= qm \end{aligned}$$

for some integer q. Further, because $g \in G$, $0 \leq g < m$, hence $q = 0, 1$.

$$\begin{aligned} q = 0 &\Rightarrow g = 0 \\ q = 1 &\Rightarrow 2g = m \\ &\Rightarrow 2|m \end{aligned}$$

Both are contradictions, as $g \neq 0 \ mod \ m$, and m is odd. Hence G does not contain an element $g \neq e$ such that $g \diamond g = e$ when m is an odd number.

c. (12 points DIFFICULT, extra credit) If $m = p$, where p is prime, will G have a non-trivial subgroup? If so, provide it. If not, say why. If you cannot do this for the general case, do it for the specific case of $m = 7$ for half credit.

Solution: No, G will not have a non-trivial subgroup. We will prove this by contradiction.

Suppose G is a non-trivial subgroup. Then it contains the element $k \neq 0 \ mod \ m$. k is relatively prime to p. Hence $\exists A, B \in \mathbb{Z}$ such that $Ak + Bp = 1$, or $Ak = 1 \ mod \ p$. If $A > 0$, k is added to itself A times to get 1, hence 1 is in the non-trivial subgroup which is closed under addition $mod \ m$. Hence, all other elements in G are in the subgroup, which is hence not non-trivial. If $A < 0$, $-k \ mod \ m$ is added to itself A times to get 1, and, because subgroups contain inverses, $-k \ mod \ m$ is in the subgroup, and, again, the subgroup is not non-trivial.

Half credit for a convincing explanation for $m = 7$, which eliminates all possible non-trivial subgroups.