CSCI 124 - Discrete Structures II - Spring 2006
George Washington University

Extra Credit HW
due 6 December 2007, by 6 pm

Policy on collaboration: All examinations, papers, and other graded work products and assignments are to be completed in conformance with The George Washington University Code of Academic Integrity. You may discuss HWs among yourselves, and work on them in groups. However, each student is expected to write his or her own HW out independently; you may not copy one another's assignments, even in part. You may not collaborate with others on the test and final.

You are expected to cite all your sources in any written work that is not closed book: papers, books, web sites, discussions with others - faculty, friends, students. For example, if, in a group, one student has a major idea that leads to a solution to a HW problem, all other students in the group should cite this student.

Any violations will be treated as violations of the Code of Academic Integrity.

1. (7 points) Construct a subset of the two-dimensional vector space over $\mathbb{R}$ that is closed under scalar multiplication, but not under vector addition. Justify your answer.

2. (7 points each) Which of the following subsets of $\mathbb{R}$ are subspaces (provide careful justification of your answer):

a. The plane of vectors $b = (b_1, b_2, b_3)$ with first component $b_1 = 0$.

b. The plane of vectors $b = (b_1, b_2, b_3)$ with first component $b_1 = c \neq 0$.

c. The vectors $b = (b_1, b_2, b_3)$ with $b_1 \times b_2 = 0$ (that is, the product of b_1 and b_2 is zero).

d. The solitary vector $b = (0, 0, 0)$

e. All combinations of two given vectors, $x = (1, 1, 0)$ and $y = (2, 0, 1)$

f. The vectors $b = (b_1, b_2, b_3)$ that satisfy $b_3 - b_2 + 3b_1 = 0$.

3A. Consider a square with vertices a, b, c, d. Consider the set of clockwise rotations of the square by multiples of $90°$: $g_1 : 90°$, $g_2 : 180°$, $g_3 : 270°$, $g_4 : 360°$ (see Figure 1).

Let $G = \{g_1, g_2, g_3, g_4\}$. Define an operation on elements of G as follows: $g_i \diamond g_j$ is the rotation obtained by performing first g_i and then g_j. For example, $g_1 \diamond g_2 = g_3$, see Figure 2.

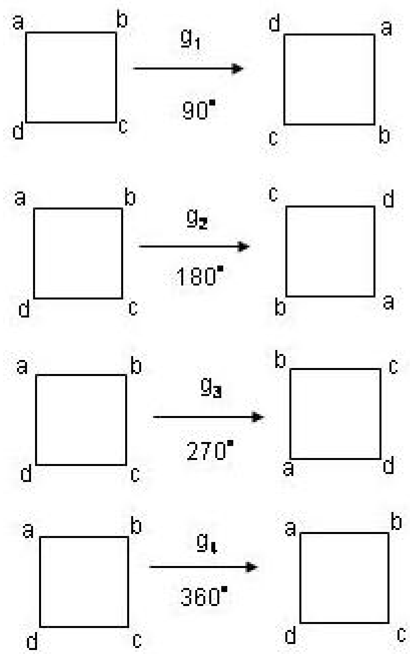

Figure 1: Rotations

It is known that G is a group under this operation. You do not need to prove this. Answer the following:

1. (5 points) What is the identity element of the group? Why?

2. (5 points) Which element g is such that $g \diamond g = e$ where e is the identity?

3. (10 points) Recall that a subgroup of G is a subset of G that satisfies all the group conditions with respect to $\diamond$. The trivial subgroups of G are $\{e\}$ and G. Does G contain a non-trivial subgroup? If not, why? If it does, what is it?

B. Consider an isosceles triangle, and the group consisting of its rotations, clockwise, by multiples of 120°.

1. (10 points) Does this group contain a non-trivial subgroup? Why or why not?

2. (5 points) Does this group contain an element g such that $g \diamond g = e$?

3. (10 points) Consider a regular n-gone: an n-sided polygon with all sides equal. A rotation by $\frac{2\pi}{n}^\circ$ moves one vertex to the next one. Consider the group of rotations of all multiples of $\frac{2\pi}{n}^\circ$. For what values of n will this group contain an element g such that $g^2 = e$?

4. Consider the group:

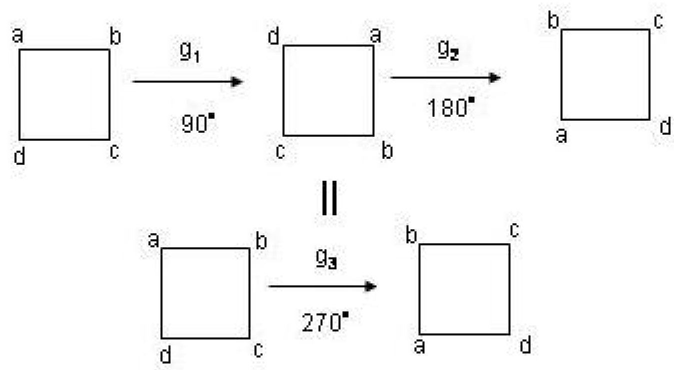

Figure 2: $g_1 \diamond g_2$

- $G = \mathbb{Z}_m$
- group operation $\diamond$ is addition *modulo* m

e denotes the identity in G.

Recall that a subgroup of G is a subset of G that satisfies all the group conditions with respect to $\diamond$. The trivial subgroups of G are $\{e\}$ and G.

You may not use Lagrange's theorem for any part of this question as it has not been covered in class.

a. (3 points) Provide an element $g \neq e$ of G such that $g \diamond g = e$ when $m = 6$.

b. (5 points) Show that G does not contain an element $g \neq e$ such that $g \diamond g = e$ when m is an odd number.

c. (12 points DIFFICULT, extra credit) If $m = p$, where p is prime, will G have a non-trivial subgroup? If so, provide it. If not, say why. If you cannot do this for the general case, do it for the specific case of $m = 7$ for half credit. (You have one more page for deciding midway that you want to do the special example. You will be given credit only for one of the two: the general or the specific case).