Vector Spaces: Notes for CSci 124

Poorvi L. Vora

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1 Definition [2, 1]

A vector space over a field F (for the moment we consider only the field of real numbers, $\mathbb{R}$) is a set of vectors $\mathbf{V}$ with two operations:

vector addition: $\mathbf{V} \times \mathbf{V} \rightarrow \mathbf{V}$ denoted $\mathbf{v} + \mathbf{w}$ for $\mathbf{v}$ and $\mathbf{w} \in \mathbf{V}$

and

scalar multiplication: $F \times \mathbf{V} \rightarrow \mathbf{V}$ denoted $c\mathbf{v}$ for $c \in F$ and $\mathbf{v} \in \mathbf{V}$

such that:

1. $\mathbf{V}$ is closed with respect to (wrt) vector addition. That is, $\mathbf{v}, \mathbf{w} \in \mathbf{V} \Rightarrow \mathbf{v} + \mathbf{w} \in \mathbf{V}$.
2. $\mathbf{V}$ is closed wrt scalar multiplication. That is, $c \in F, \mathbf{v} \in \mathbf{V} \Rightarrow c\mathbf{v} \in \mathbf{V}$
3. The operation of vector addition has the following properties:
   (a) It is *commutative*. That is, $\mathbf{v}, \mathbf{w} \in \mathbf{V} \Rightarrow \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$
   (b) It is *associative*. That is, $\mathbf{v}, \mathbf{w}, \mathbf{x} \in \mathbf{V} \Rightarrow (\mathbf{v} + \mathbf{w}) + \mathbf{x} = \mathbf{v} + (\mathbf{w} + \mathbf{x})$
   (c) It has an *identity*. That is, $\exists\, \mathbf{e} \in \mathbf{V}$ such that, $\forall\, \mathbf{v} \in \mathbf{V}, \mathbf{v} + \mathbf{e} = \mathbf{v}$.
   $\mathbf{e}$ is often denoted $\mathbf{0}$, the zero vector.
   (d) It has an *inverse*. That is, $\forall\, \mathbf{v} \in \mathbf{V},\ \exists\, (-\mathbf{v}) \in \mathbf{V}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{e}$.
4. The operation of scalar multiplication has the following properties:
   (a) It has an *identity* that is also the identity of F, denoted 1. That is, $\exists\, 1 \in F$ such that, $\forall\, \mathbf{v} \in \mathbf{V}, 1\mathbf{v} = \mathbf{v}$, and, $\forall\, c \in F, 1c = c$.
   (b) It is consistent with field multiplication. That is, $\mathbf{v} \in \mathbf{V}, c, d, \in F \Rightarrow (cd)\mathbf{v} = c(d\mathbf{v})$
   (c) It is distributive over vector addition. That is, $c \in F, \mathbf{v}, \mathbf{w} \in \mathbf{V} \Rightarrow c(\mathbf{v} + \mathbf{w}) = c\mathbf{v} + c\mathbf{w}$.
   (d) It is distributive over field addition. That is, $\mathbf{v} \in \mathbf{V}, c, d, \in F \Rightarrow (c + d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$

2 Examples of Vector Spaces

1. The two-dimensional cartesian space over the field of real numbers, $\mathbb{R}$.
2. The three-dimensional cartesian space over $\mathbb{R}$.
3. The n-dimensional cartesian space over $\mathbb{R}$, for positive integer n.
4. The complex numbers ($\mathbb{C}$) over $\mathbb{R}$. Notice that the definition of a vector space says nothing about multiplication of two vectors, or the magnitude of the vector, or its conjugate. These are all associated with $\mathbb{C}$, but are not related to it being a vector space.
5. $\mathbb{R}$ over $\mathbb{R}$.
6. The set of $n \times m$ matrices for positive integers n and m over $\mathbb{R}$.

3 Examples of the Applications of Ideas from Vector Spaces

Now consider a two-dimensional grey scale image of size $m \times n$, with each pixel taking on values from 0 to 255, inclusive. For example, consider the following image of size 3×4:

Figure 1: A grey-scale image

It can also be represented as an array of its pixel values:

100	200	225	235
125	225	250	155
150	250	225	125

The grey-scale values are often piled into a single column, row by row. That is, the image is often represented as a single column of size mn, with the first n values being the first row, in order, the next n values the second row, in order, and so on. In our example, the image would be represented as:

$$\begin{bmatrix} 100 \\ 200 \\ 225 \\ 235 \\ 125 \\ 225 \\ 250 \\ 155 \\ 150 \\ 250 \\ 225 \\ 125 \end{bmatrix}$$

One often needs to process these images – for example, to deblur or compress them. That is, one wants to perform mathematical operations on these images. It is hence useful to think of these images as being contained in a vector space of dimension mn (in our example, the dimension is 12) over the real numbers. Addition is defined as regular component-wise addition, and scalar multiplication is defined as multiplication of each pixel value by the scalar.

Now notice that the set of images is, by itself, not a vector space, for several reasons. First, it is not closed wrt field multiplication (take an image that is constant: all pixel values are 200; multiplying it by 2 results in a vector that does not correspond to a grey-scale image, because each component is larger than 255). Second, it is not closed wrt vector addition (take two images, each constant with pixel value 200; adding them results in a vector that does not correspond to a grey-scale image, because each component is larger than 255). Third, the additive inverse of a non-zero image is not among the images, because at least one of its pixel values is negative.

However, one may think of the images as all lying within the vector space during processing. All the processing is performed in the vector space. At the end of the processing, a resulting vector of size mn is obtained. Each of the components of the resulting vector may not be an integer between 0 and 255; that is, the resulting vector may not correspond to a grey-scale image. The resulting vector is transformed into the closest vector that does represent a grey-scale image, and this is the image resulting from the processing.

In our example, we may wish to perform averaging: replace each pixel value with the average pixel value in the 3×3 window around (and including) the pixel. So, for example, the pixel in the second row and second column of our example, of value 225, will be replaced by one of value

$$\frac{100+200+225+125+225+250+150+250+225}{9} = \frac{1750}{9}$$

rounded off to 194.

Each of the following may also be treated as vectors in a larger vector space:

1. The three color values at a single pixel in a color image.

2. An $n \times m$ color image, which has $3nm$ components (3 colors for each pixel).
3. A video clip consisting of p frames, each a color frame of size $n \times m$. This will have $3pnm$ components (each frame has $3nm$ components).
4. n consecutive samples of a sound signal, taken Δt apart.

4 Example of a Set that is not a Vector Space

Example 1: The first quadrant in two-dimensional euclidean space over the field of real numbers, $\mathbb{R}$.

$$\mathbf{A} = \{\mathbf{b} = (x, y) | x, y \in \mathbb{R}, x \geq 0, y \geq 0\}$$

(The symbol "|" stands for "such that").

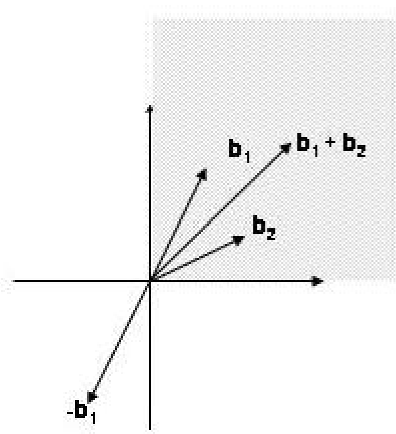

Figure 2: The first quadrant is not a vector space

Is **A** closed wrt vector addition?

Take any two vectors from **A**, (x_1, y_1) and (x_2, y_2). Their sum is $(x_1 + x_2, y_1 + y_2)$. This is in **A** if and only if $x_1 + x_2, y_1 + y_2 \in \mathbb{R}$, and $x_1 + x_2 \geq 0, y_1 + y_2 \geq 0$

- By the definition of **A**, $x_1, y_1, x_2, y_2 \in \mathbb{R}$. This implies that $x_1 + x_2, y_1 + y_2 \in \mathbb{R}$.
- Also by the definition of **A**, $x_1 \geq 0, y_1 \geq 0, x_2 \geq 0, y_2 \geq 0$. This implies that $x_1 + x_2 \geq 0, y_1 + y_2 \geq 0$. (See Figure 2).

Hence **A** is closed wrt vector addition.

Is **A** closed wrt scalar multiplication?

Take any vector (x, y) from $\mathbf{A}$, and any scalar c from $\mathbb{R}$. Then $c(x, y) = (cx, cy)$. This is in $\mathbf{A}$ if and only if $cx, cy \in \mathbb{R}$, and $cx \geq 0, cy \geq 0$.

- By the definition of $\mathbf{A}$, $x, y \in \mathbb{R}$. This implies that $cx, cy \in \mathbb{R}$ (product of two real numbers is real).
- Also by the definition of $\mathbf{A}$, $x \geq 0, y \geq 0$. If $c \geq 0$, then $cx \geq 0$, $cy \geq 0$. However, if $c < 0$, $cx \leq 0$, $cy \leq 0$. In particular, for any $x, y \neq 0$ and $c < 0$, $c(x, y)$ is not in $\mathbf{A}$. For example, if $c = -1$, and $\mathbf{b} \in \mathbf{A}$, but $\mathbf{b} \neq (0, 0)$, $c\mathbf{b}$ is in the third quadrant.

Hence $\mathbf{A}$ is not closed wrt scalar multiplication.

Hence $\mathbf{A}$ is not a vector space.

5 Subspaces

Definition A subset $\mathbf{A}$ of vector space $\mathbf{V}$ is a *subspace* of $\mathbf{V}$ if it is closed wrt vector addition and scalar multiplication.

Examples: The x and y axes are subspaces of $\mathbb{R}^2$ as is any straight line passing through the origin.

Example 2: Consider a straight line:

$$\mathbf{A} = \{\mathbf{b} = (x, y) | y = mx + c\}$$

where m and c are real constants.

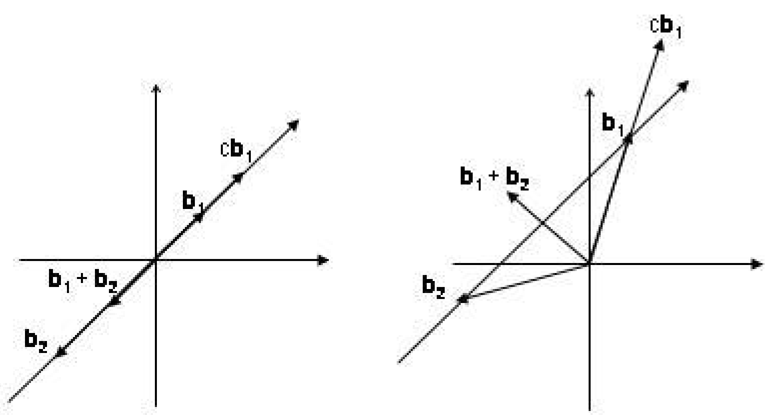

Figure 3: A straight line is a subspace of $\mathbb{R}^2$ if and only if it passes through the origin

Consider any two vectors in $\mathbf{A}$: $\mathbf{b}_1 = (x_1, y_1)$ and $\mathbf{b}_2 = (x_2, y_2)$. Then $\mathbf{b}_1 + \mathbf{b}_2 = (x_1 + x_2, y_1 + y_2)$. Is $\mathbf{b}_1 + \mathbf{b}_2$ in $\mathbf{A}$? For this, we need to determine if $\mathbf{b}_1 + \mathbf{b}_2$ is on the line, that is if

$$y_1 + y_2 = m(x_1 + x_2) + c \tag{1}$$

We know that $y_1 = mx_1 + c$ and $y_2 = mx_2 + c$. Hence

$$y_1 + y_2 = m(x_1 + x_2) + 2c \tag{2}$$

Equation (2) is equation (1) if and only if $c = 0$. That is, $\mathbf{A}$ is closed under vector addition if and only if $c = 0$, i.e. the line passes through the origin. (See Figure 3).

Now consider scalar multiplication. Let $\alpha \in \mathbb{R}$. Let $\mathbf{b} = (x, y) \in \mathbf{V}$. Then $\alpha\mathbf{b} = (\alpha x, \alpha y)$. This is in $\mathbf{A}$ if

$$\alpha y = \alpha m x + c \tag{3}$$

In this case,

$$y = mx + c \Rightarrow \alpha y = \alpha m x + \alpha c \tag{4}$$

Equation (4) is equation (3) if and only if $c = 0$. That is, $\mathbf{A}$ is closed under vector addition if and only if $c = 0$, i.e. the line passes through the origin. (See Figure 3).

Hence any straight line through the origin is a subspace of $\mathbb{R}^2$, but a straight line not through the origin is not.

6 Linear Combinations of Vectors and Spans

Definition A *linear combination* of n vectors $\mathbf{v}_i \in \mathbf{V}$ is a vector $\sum_{i=1}^{n} a_i \mathbf{v}_i$ for some $\{a_i\}_{i=1}^{n} \subset F$.

The values a_i above are often referred to as the *coefficients*.

Example 3:

$$2\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} - 7\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ -2 \\ 12 \end{bmatrix}$$

is a linear combination of

$$\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

A shorthand way of saying that a vector $\mathbf{v}$ is a linear combination of $\{\mathbf{v}_i\}_{i=1}^{n}$ is to say it lies in the *span* of these vectors. The *span* is the set of all possible linear combinations.

Definition The *span* of a finite set of vectors $\mathbf{B} = \{\mathbf{v}_i\}_{i=1}^{n} \subset \mathbf{V}$ is

$$Span(\mathbf{B}) = \{\sum_{i=1}^{n} a_i \mathbf{v}_i | a_i \in F\}$$

How does one determine if a given vector **v** belongs to the span of $\{\mathbf{v}_i\}_{i=1}^n$? Further, if it does, how does one determine the values a_i? Answer: solve a set of linear equations!

6.1 Determining if $\mathbf{v} \in Span(\{\mathbf{v}_i\}_{i=1}^n)$

1. The question is, are there values a_1, a_2, ..., a_n such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_n\mathbf{v}_n = \mathbf{v}$$

2. Form a matrix **A** with $\{\mathbf{v}_i\}_{i=1}^n$ as its columns

$$\mathbf{A} = [\mathbf{v}_1\mathbf{v}_2....\mathbf{v}_n]$$

3. Write a vector **a** containing the unknown coefficients a_i:

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ . \\ a_n \end{bmatrix}$$

4. Solve the set of linear equations $\mathbf{A}\mathbf{a} = \mathbf{v}$ using gaussian elimination

5. If there is exactly one solution, or many, $\mathbf{v} \in Span(\{\mathbf{v}_i\}_{i=1}^n)$, and the solutions provide the coefficients.

6. If there are no solutions, **v** is not in $Span(\{\mathbf{v}_i\}_{i=1}^n)$

6.2 Examples

Example 4: Determine if

$$\mathbf{v} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

belongs to $Span(\{\mathbf{v}_i\}_{i=1}^3)$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

We would proceed as follows. The question being asked is: are there values a_1, a_2, and a_3 such that

$$a_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

That is, does the set of equations:

$$\begin{array}{ccccccc} a_1 & + & a_2 & + & a_3 & = & 2 \\ a_1 & - & a_2 & + & a_3 & = & -2 \\ a_1 & + & a_2 & - & a_3 & = & 3 \end{array}$$

have a solution?

We can solve this using gaussian elimination right away:

$$\begin{array}{ccccccc} a_1 & + & a_2 & + & a_3 & = & 2 \\ & - & 2a_2 & & & = & -4 \\ & & & - & 2a_3 & = & 1 \end{array}$$

which gives us: $a_2 = 2$, $a_3 = -\frac{1}{2}$, and $a_1 = \frac{1}{2}$. Which means a solution does exist, and the given vector **v** may be expressed as a linear combination of the vectors $\{\mathbf{v}_i\}_{i=1}^3$.

We could also have written the above equation as a matrix equation. Recall, we did this when we were solving linear equations. The matrix is constructed with the vectors $\mathbf{v}_i$ as columns ($\mathbf{v}_1$ is the first column of the matrix, $\mathbf{v}_2$ is the second, and so on):

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

The above set of equations can be solved by trying to invert the matrix if it is square, or, as above, by using gaussian elimination, which is usually the quickest and cleanest approach.

In the above example, $\mathbf{v} \in Span(\mathbf{B})$. What happens when **v** *does not* belong to $Span(\{\mathbf{v}_i\}_{i=1}^n)$? Or when it does, but there are many possible values of $a_1, a_2, ...a_n$? Answer: the same thing that happens when a set of equations has no solutions, or has many solutions, respectively.

Example 5: Determine if

$$\begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

belongs to $Span(\{\mathbf{v}_i\}_{i=1}^3)$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

If so, what are the coefficients? In matrix form, the question is, find a solution to:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

if it exists. If you proceed with gaussian elimination, you will be fine. If you try to invert the matrix, you won't be successful, because its determinant is zero. So always first attempt gaussian elimination. Don't forget to perform the same operations on the right-hand side.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$$

This gives you a final equation of the form $0 = 1$. This is an inconsistent set of equations, which means that **v** is not in $Span\{\mathbf{v}_i\}_{i=1}^3$.

Example 6: Determine if

$$\begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

is a linear combination of

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{5}$$

If so, what are the coefficients? Again, in matrix form, the question is, find a solution to:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}$$

if it exists. Gaussian elimination gives:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

where the final equation is of the form $0 = 0$. This means you have a consistent set of equations with several solutions. One solution is $a_3 = 1$, $a_2 = 0$, $a_1 = 3$. That is,

$$3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix} \tag{6}$$

6.3 Discussion

In Example 6, three equations in three unknowns gave several solutions. In this case, one of the equations was redundant: it provided no new information, which is why it disappeared on gaussian elimination. There were essentially two equations in three unknowns, and hence several solutions.

We've seen this before while studying linear equations, but we see it in a different light now: one of the equations is redundant because one of the matrix columns (or vectors $\mathbf{v}_i$) is redundant. That is, any one of the three vectors $\{\mathbf{v}_i\}_{i=1}^3$ lies in the span of the other two, and hence a third vector does not provide an additional, independent direction. For example, of the three vectors in equation (5),

$$\frac{1}{2}\begin{bmatrix}1\\1\\1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}1\\-1\\1\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix}$$

That is, one can add some amounts of the vectors to cancel one another out, and get the zero vector. For example,

$$\frac{1}{2}\begin{bmatrix}1\\1\\1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}1\\-1\\1\end{bmatrix}-\begin{bmatrix}0\\1\\0\end{bmatrix}=\mathbf{0}=\begin{bmatrix}0\\0\\0\end{bmatrix} \tag{7}$$

We can add any amounts of the zero vector (equation (7)) to equation (6) to get several solutions to our problem. Suppose we add the zero vector n times, we get equation (6) + $n\times$ equation (7):

$$3\begin{bmatrix}1\\1\\1\end{bmatrix}+0\begin{bmatrix}1\\-1\\1\end{bmatrix}+1\begin{bmatrix}0\\1\\0\end{bmatrix}+n(\frac{1}{2}\begin{bmatrix}1\\1\\1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}1\\-1\\1\end{bmatrix}-\begin{bmatrix}0\\1\\0\end{bmatrix})=\begin{bmatrix}3\\4\\3\end{bmatrix}$$

or

$$(3+\frac{n}{2})\begin{bmatrix}1\\1\\1\end{bmatrix}-\frac{n}{2}\begin{bmatrix}1\\-1\\1\end{bmatrix}+(1-n)\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}3\\4\\3\end{bmatrix}$$

Hence, in general, $a_1 = 3+\frac{n}{2}$, $a_2 = -\frac{n}{2}$, $a_3 = 1-n$ for all integers n.

The above issue does not arise when the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are *linearly independent*. Loosely speaking, a set of vectors $\{\mathbf{v}_i\}_{i=1}^n$ is linearly independent if the only way of getting the zero vector by combining them is to take a zero amount of each vector.

7 Linear Independence

Definition A set of vectors $\{\mathbf{v}_i\}_{i=1}^n \subset \mathbf{V}$ is *linearly independent* if $\sum_{i=1}^n a_i\mathbf{v}_i = \mathbf{0} \Leftrightarrow a_i = 0\ for\ i = 1,2,...n$

Definition A set of vectors $\{\mathbf{v}_i\}_{i=1}^n \subset \mathbf{V}$ that is not linearly independent is *linearly dependent*.

Example 7: The vectors:

$$\mathbf{v}_1=\begin{bmatrix}2\\2\\3\end{bmatrix},\mathbf{v}_2=\begin{bmatrix}1\\3\\4\end{bmatrix},\mathbf{v}_3=\begin{bmatrix}3\\5\\7\end{bmatrix}$$

are linearly dependent because $\mathbf{v}_3 = \mathbf{v}_1+\mathbf{v}_2$. That is, because $\mathbf{v}_1+\mathbf{v}_2-\mathbf{v}_3 = \mathbf{0}$. Hence, $\sum_{i=1}^3 a_i\mathbf{v}_i = \mathbf{0}$ does not imply that $a_i = 0\ for\ i = 1,2,3$.

Example 8: The vectors we saw earlier,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

are also linearly dependent because

$$\frac{1}{2}\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$$

and $\sum_{i=1}^{3} a_i \mathbf{v}_i = \mathbf{0}$ does not imply that $a_i = 0 \; for \; i = 1, 2, 3$.

Example 9: On the other hand, the vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

are linearly independent because $\sum_{i=1}^{3} a_i \mathbf{v}_i = \mathbf{0}$ implies that $a_i = 0 \; for \; i = 1, 2, 3$.

In general, how would one determine if a set of vectors was linear independent? Again, linear equations!

7.1 Determining the Linear Independence of a Set of Vectors $\{\mathbf{v}_i\}_{i=1}^{n}$

1. The question is, are there values a_1, a_2, ..., a_n, not all zero, such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_n\mathbf{v}_n = \mathbf{0}$$

2. Form a matrix $\mathbf{A}$ with $\{\mathbf{v}_i\}_{i=1}^{n}$ as its columns

$$\mathbf{A} = [\mathbf{v}_1 \mathbf{v}_2 .... \mathbf{v}_n]$$

3. Write a vector **a** containing the unknown coefficients a_i:

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ . \\ a_n \end{bmatrix}$$

4. Solve the set of linear equations $\mathbf{Aa} = \mathbf{0}$ using gaussian elimination. How many solutions does it have?

(a) We know that there is one solution for these equations, it is $\mathbf{a} = 0$.

(b) If there is exactly one solution, and no more, the vectors $\{\mathbf{v}_i\}_{i=1}^n$ are linearly independent.

(c) If there are several solutions, the vectors $\{\mathbf{v}_i\}_{i=1}^n$ are not linearly independent

7.2 Example

Example 10: Is the set of vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

linearly independent? In other words, the question is, are there values a_1, a_2, a_3 and a_4, not all zero, such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 = \mathbf{0}$$

or

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \mathbf{0}$$

Now, the above are four equations is four unknowns, for which we already know one solution: $a_1 = a_2 = a_3 = a_4 = 0$. So the set of vectors is linearly independent only if there are no more solutions. Gaussian elimination, as always.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \mathbf{0}$$

Swap for a non-zero pivot. Remember to swap rhs as well, though, in this case, because the rhs is zero, it doesn't matter.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \mathbf{0}$$

Because the final equation contains a single unknown, there is only one solution to this set of equations. We know that $a_1 = a_2 = a_3 = a_4 = 0$ is *a* solution, now we know it is the only solution. (You can check that the gaussian elimination gives you that solution: $-4a_4 = 0$, or $a_4 = 0$. Working backwards, you get that all four unknowns are zero.)

We could have determined the same by checking if the matrix corresponding to the equations was invertible. If it was, then there would be exactly one solution. We didn't invert the matrix because this works only when there are n equations in n unknowns, which is not always the case.

7.3 Spans and Linear Equations

Consider a set of n equations in m unknowns. We have observed examples of the following facts:

1. The set of equations $\mathbf{Ax} = \mathbf{b}$ has *a solution* (that is, at least one solution) if $\mathbf{b} \in Span\{\mathbf{A}_i\}_{i=1}^m$ where $\mathbf{A}_i$ represents the i^{th} column of $\mathbf{A}$. That is, the equations have a solution if **b** can be expressed as a linear combination of $\{\mathbf{A}_i\}_{i=1}^m$.
2. The converse is also true: the set if equations has no solution if **b** is not in $Span\{\mathbf{A}_i\}_{i=1}^m$.
3. If $\mathbf{b} \in Span\{\mathbf{A}_i\}_{i=1}^m$ (the set of equations has at least one solution), then:
   (a) The set of equations has more than one solution if $\{\mathbf{A}_i\}_{i=1}^m$ are not linearly independent.
   (b) The converse is also true. The set of equation has only one solution if $\{\mathbf{A}_i\}_{i=1}^m$ are linearly independent.

8 Bases of a Vector Space

Consider the vector space $\mathbb{R}^2$. Consider any vector in it. This vector can be written as a linear combination of

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x\mathbf{v}_1 + y\mathbf{v}_2$$

That is

$$\mathbb{R}^2 \subseteq Span(\{\mathbf{v}_1, \mathbf{v}_2\})$$

Further, there is no vector in $Span(\{\mathbf{v}_1, \mathbf{v}_2\})$ that is not in $\mathbb{R}^2$. Hence

$$\mathbb{R}^2 = Span(\{\mathbf{v}_1, \mathbf{v}_2\})$$

and we say that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a *basis* for $\mathbb{R}^2$. A *basis* of a vector space is a set of linearly independent vectors whose span is the vector space.

Definition A set of vectors $\mathbf{B} \subset \mathbf{V}$ is a basis for $\mathbf{V}$ if and only if $\mathbf{B}$ is linearly independent, and $Span(\mathbf{B}) = \mathbf{V}$.

In general, the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ . \\ . \\ . \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}, ...\mathbf{v}_i = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \\ . \\ . \\ . \\ 0 \end{bmatrix}, ...\mathbf{v}_n = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ . \\ . \\ . \\ 1 \end{bmatrix}$$

form a basis for $\mathbb{R}^n$, known as the *standard basis*.

Other bases exist, see Example 11 and Exercises 4 and 5 for examples.

8.1 Summary of Methodology to Determine if a set A is a Basis for a Vector Space V

1. Determine if the set **A** is linearly independent. If it is not, it is not a basis.

2. Determine if every vector in **V** can be expressed as a linear combination of the vectors in **A**. If so, you have shown that $\mathbf{V} \subseteq Span(\mathbf{A})$

3. Determine if every vector in $Span(\mathbf{A})$ belongs to **V**. If so, you have shown that $Span(\mathbf{A}) \subseteq \mathbf{V}$.

   In the problems we address, this will be easy. **V** will be one of $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$ or $\mathbb{R}^4$. As the vectors in **A** will be either one, two, three or four dimensional, it will be straightforward to say whether these vectors, and any linear combinations of them, belong to the $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$ or $\mathbb{R}^4$. See Example 11, and Exercises 4 and 5 for further clarification.

4. If 2 and 3 above hold, we have $\mathbf{V} = Span(\mathbf{A})$

5. If 1 and 4 above hold, **A** is a basis for **V**

8.2 Example

Example 11:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

is a basis for $\mathbb{R}^4$.

We have already shown the above set is linearly independent (see Example 10). To show that it is a basis, we also need to show that $\mathbb{R}^4$ is its span. For this, we need to consider any vector in $\mathbb{R}^4$:

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

and show that it is a linear combination of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$. That is, we need to show that there exist a_1, a_2, a_3, a_4 such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

or

$$a_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} + a_4 \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

Again, the equations may be written as:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

which may be reduced:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} w \\ x-w \\ y-w \\ z-w \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} w \\ y-w \\ x-w \\ z-w \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} w \\ y-w \\ x-w \\ z-w-(y-w)=z-y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} w \\ y-w \\ x-w \\ z-y+x-w \end{bmatrix}$$

$a_4 = \frac{w-x+y-z}{4}$, $a_3 = \frac{w-x-y+z}{4}$, $a_2 = \frac{w+x-y-z}{4}$, $a_1 = \frac{w+x+y+z}{4}$

Thus, given any vector in $\mathbb{R}^4$, we have obtained its coefficients wrt $\{\mathbf{v}_i\}_{i=1}^4$, hence $\mathbb{R}^4 \subseteq Span(\{\mathbf{v}_i\}_{i=1}^4)$.

Further, because each of the $\mathbf{v}_i$ is a four dimensional column, linear combinations of the $\mathbf{v}_i$ cannot lie outside the four-dimensional space over the real numbers. Hence, $Span(\{\mathbf{v}_i\}_{i=1}^4) \subseteq \mathbb{R}^4$.

Hence $Span(\{\mathbf{v}_i\}_{i=1}^4) = \mathbb{R}^4$. Example 10 shows that $\{\mathbf{v}_i\}_{i=1}^4$ are linearly independent, hence $\{\mathbf{v}_i\}_{i=1}^4$ forms a basis for $\mathbb{R}^4$.

8.3 The Dimension of a Vector Space

Definition The number of vectors in a finite basis of a vector space is its *dimension*.

Fact: The number of vectors in a basis for $\mathbb{R}^n$ is n.

This gives us one additional fact connecting linear equations to vector spaces. Consider a set of n equations in n unknowns, represented by the matrix equation $\mathbf{Ax} = \mathbf{b}$. Notice that $\mathbf{A}$ is square. We now add to the observations of section 7.3 for the special case of the square matrix.

1. The equations have exactly one solution if **A** is invertible, or, equivalently, if all its columns are independent, or, equivalently, if the columns form a basis for $\mathbb{R}^n$.

9 Eigenvalues and Eigenvectors

Definition: λ is an *eigenvalue* of square matrix $\mathbf{A}$ if $\exists$ non-zero vector **x** such that $\mathbf{Ax} = \lambda\mathbf{x}$.

x is termed an *eigenvector*.

Notice that, if $\mathbf{Ax} = \lambda\mathbf{x}$, then $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ where $\mathbf{I}$ is the identity matrix of the size of matrix $\mathbf{A}$. As $\mathbf{x} \neq \mathbf{0}$, this means that $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$ has many solutions, and hence that $(\mathbf{A} - \lambda\mathbf{I})$ is not invertible, or that its determinant is zero.

9.1 Determining Eigenvectors and Eigenvalues

To determine the eigenvalues and corresponding eigenvectors of a matrix $\mathbf{A}$ do the following:

1. Solve the following equation for λ: $|\mathbf{A} - \lambda\mathbf{I}| = 0$
2. Using the values of λ obtained, determine a vector **x** (for each λ) such that $\mathbf{Ax} = \lambda\mathbf{x}$

9.2 Example

Example 12: Find the two eigenvalues and an eigenvector for each eigenvalue of **A** where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Answer:

$$\begin{vmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 - 1 = 0$$

$$\Rightarrow (\lambda+1)(\lambda-1) = 0$$

$$\Rightarrow \lambda = +1, -1$$

$\lambda = +1$ and $\lambda = -1$ are the two eigenvalues. To find the corresponding eigenvectors, find $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{Ax} = \lambda\mathbf{x}$:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For $\lambda = 1$:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$x_1 = x_2$ is the solution. So vectors of the form $(1, 1)$, $(2, 2)$ .... are eigenvectors corresponding to $\lambda = 1$. For $\lambda = -1$:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -x_2 \end{bmatrix}$$

$x_1 = -x_2$ is the solution. So vectors of the form $(1, -1)$, $(2, -2)$ .... are eigenvectors corresponding to $\lambda = -1$.

10 Exercises

1. Consider the union of the first and third quadrants in two-dimensional euclidean space over the field of real numbers, $\mathbb{R}$.

$$\mathbf{A} = \{b = (x, y) | x, y \in \mathbb{R}, x \geq 0, y \geq 0\} \cup \{b = (x, y) | x, y \in \mathbb{R}, x \leq 0, y \leq 0\}$$

Is **A** closed wrt vector addition? Is it closed under scalar multiplication? Is it a vector space?

2. Is the set of integers, $\mathbb{Z}$, a vector space over $\mathbb{R}$?

3. Which of the following sets of vectors is linearly independent? a.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

b.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

c.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

4. Show that

$$\mathbf{w}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

is a basis for $\mathbb{R}^3$.

5. Show that

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

is a basis for $\mathbb{R}^3$.

6. Express the vectors in (a)-(d) as linear combinations of the set of vectors:

$$\mathbf{w}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

a.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

b.

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

c.

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

d.

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

7. Find the eigenvalues and corresponding eigenvectors of the following matrices:

a.

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

b.

$$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$$

Solutions

1. Consider the union of the first and third quadrants in two-dimensional euclidean space over the field of real numbers, $\mathbb{R}$.

$$\mathbf{A} = \{b = (x, y)|x, y \in \mathbb{R}, x \geq 0, y \geq 0\} \cup \{b = (x, y)|x, y \in \mathbb{R}, x \leq 0, y \leq 0\}$$

Is **A** closed wrt vector addition? Is it closed under scalar multiplication? Is it a vector space?

Answer: Is **A** closed wrt vector addition?

Take any two vectors from **A**, (x_1, y_1) and (x_2, y_2). Their sum is $(x_1 + x_2, y_1 + y_2)$. This is in **A** if and only if

(a) $x_1 + x_2, y_1 + y_2 \in \mathbb{R}$, and

(b) $x_1 + x_2 \geq 0, y_1 + y_2 \geq 0$ OR $x_1 + x_2 \leq 0, y_1 + y_2 \leq 0$

By the definition of **A**, $x_1, y_1, x_2, y_2 \in \mathbb{R}$. This implies that $x_1 + x_2, y_1 + y_2 \in \mathbb{R}$.

Also by the definition of **A**,

either $x_1 \geq 0, y_1 \geq 0$ OR $x_1 \leq 0, y_1 \leq 0$,

and

either $x_2 \geq 0, y_2 \geq 0$ OR $x_2 \leq 0, y_2 \leq 0$.

We cannot say that $x_1 + x_2 \geq 0, y_1 + y_2 \geq 0$. For example, if $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (-3, -2)$, their sum is $(-1, 1)$ is not in **A**.

Hence **A** is not closed wrt vector addition.

Is **A** closed wrt scalar multiplication?

Take any vector (x, y) from **A**, and any scalar c from $\mathbb{R}$. Then $c(x, y) = (cx, cy)$. This is in **A** if and only if $cx, cy \in \mathbb{R}$, and either $cx \geq 0, cy \geq 0$ OR $cx \leq 0, cy \leq 0$.

By the definition of **A**, $x, y \in \mathbb{R}$. This implies that $cx, cy \in \mathbb{R}$ (product of two real numbers is real).

Also by the definition of **A**, either

(a) $x \geq 0, y \geq 0$ OR

(b) $x \leq 0, y \leq 0$.

For (a), if $c \geq 0$, then $cx \geq 0, cy \geq 0$, and $c(x, y) \in \mathbf{A}$. If $c \leq 0$, then $cx \leq 0, cy \leq 0$, and $c(x, y) \in \mathbf{A}$.

For (b), if $c \geq 0$, then $cx \leq 0, cy \leq 0$, and $c(x, y) \in \mathbf{A}$. If $c \leq 0$, then $cx \geq 0, cy \geq 0$, and $c(x, y) \in \mathbf{A}$.

Hence **A** is closed wrt scalar multiplication.

Hence **A** is not a vector space.

Recall the example of section 3 that was not a vector space: the set consisting of only the first quadrant. It was not closed under scalar multiplication, though it was closed wrt vector addition. By adding the third quadrant, we were able to construct a set closed under scalar multiplication, but now the set is not closed under vector addition.

2. Is the set of integers, $\mathbb{Z}$, a vector space over $\mathbb{R}$?

Answer: Consider any two elements of $\mathbb{Z}$. Their sum is also an integer, hence the sum of two vectors is also in the set, hence it is closed wrt vector addition.

Consider a scalar c. Multiplying it with any member of the set does not necessarily give an integer because c may not be an integer (for example, $c = \frac{1}{2}$). Hence the set is not closed wrt scalar multiplication.

Hence $\mathbb{Z}$ is not a vector space over $\mathbb{R}$.

3. Which of the following sets of vectors is linearly independent?

Answer: Try to find non-zero coefficients that combine the vectors to get **0**. If you can find the non-zero coefficients, the vectors are *not* linearly independent. Otherwise, they are.

a.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Answer: Solve

$$\mathbf{A}\mathbf{a} = \mathbf{0}$$

or

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Final equation of form $0 = 0$ hence there is more than one solution to the equations, hence the vectors are not linearly independent.

b.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Answer: Solve

$$\mathbf{Aa} = \mathbf{0}$$

or

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 1 \\ 0 & -4 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Final equation of form $0 = 0$ hence there is more than one solution to the equations, hence the vectors are not linearly independent.

c.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Answer: Solve

$$\mathbf{Aa} = \mathbf{0}$$

or

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The system has exactly one solution, $a_1 = a_2 = a_3 = a_4 = 0$. Hence the vectors are linearly independent.

4. Show that

$$\mathbf{w}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

is a basis for $\mathbb{R}^3$.

Answer: We need to show that $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$ are linearly independent, and that $Span(\{\mathbf{w}_i\}_{i=1}^3) = \mathbb{R}^3$.

First we show linear independence. Find a_1, a_2, a_3 such that:

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This system has exactly one solution, $a_1 = a_2 = a_3 = 0$, hence the vectors are linearly independent.

Next we show that $Span(\{\mathbf{w}_i\}_{i=1}^3) = \mathbb{R}^3$. First, we observe that the vectors are three dimensional, hence all combinations are also three dimensional, and $Span(\{\mathbf{w}_i\}_{i=1}^3) \subseteq \mathbb{R}^3$. Next we wish to show that $\mathbb{R}^3 \subseteq Span(\{\mathbf{w}_i\}_{i=1}^3)$. We consider any vector in $\mathbb{R}^3$, (x, y, z). We see if it can be expressed as a linear combination of $\{\mathbf{w}_i\}_{i=1}^3$.

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Gaussian elimination gives

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} x \\ z \\ y \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} x \\ z - \frac{x}{2} \\ y \end{bmatrix}$$

The solution is: $a_3 = y$, $a_2 = 2z - x$, $a_1 = x - z$. Hence $\mathbb{R}^3 \subseteq Span(\{\mathbf{w}_i\}_{i=1}^3)$.

Hence we have shown that the given vectors form a basis of $\mathbb{R}^3$.

5. Show that

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

is a basis for $\mathbb{R}^3$.

Answer: We need to show that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent, and that $Span(\{\mathbf{v}_i\}_{i=1}^3) = \mathbb{R}^3$.

First we show linear independence. Find a_1, a_2, a_3 such that:

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Has exactly one solution, $a_1 = a_2 = a_3 = 0$. Hence the vectors are linearly independent.

Next we show that $Span(\{\mathbf{v}_i\}_{i=1}^3) = \mathbb{R}^3$. First, we observe that the vectors are three dimensional, hence all combinations are also three dimensional, and $Span(\{\mathbf{v}_i\}_{i=1}^3) \subseteq \mathbb{R}^3$. Next we wish to show that $\mathbb{R}^3 \subseteq Span(\{\mathbf{v}_i\}_{i=1}^3)$. We consider any vector in $\mathbb{R}^3$, (x, y, z). We see if it can be expressed as a linear combination of $\{\mathbf{v}_i\}_{i=1}^3$.

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y \\ x \\ z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y \\ x \\ z - y \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y \\ x \\ z - y - x \end{bmatrix}$$

$a_3 = \frac{x+y-z}{2}$, $a_2 = \frac{x-y+z}{2}$, $a_1 = \frac{-x+y+z}{2}$. Hence $\mathbb{R}^3 \subseteq Span(\{\mathbf{v}_i\}_{i=1}^3)$.
Hence we have shown that the given vectors form a basis of $\mathbb{R}^3$.

6. Express the vectors in (a)-(d) as linear combinations of the set of vectors:

$$\mathbf{w}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

a.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Answer: Solve the equation:

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ \frac{5}{2} \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{5}{2} \\ 2 \end{bmatrix}$$

$a_3 = 2$, $a_2 = 5$, $a_1 = -2$

b.

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Answer: Solve the equation:

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$a_3 = 1$, $a_2 = 4$, $a_1 = -2$

c.

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Answer: Solve the equation:

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$a_3 = 1$, $a_2 = 0$, $a_1 = 0$

d.

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Answer: Solve the equation:

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$a_3 = 0$, $a_2 = 2$, $a_1 = -1$

7. Find the eigenvalues and corresponding eigenvectors of the following matrices:

a.

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Answer:

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)^2 - 1 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda - \lambda + 3 = 0$$

$$\Rightarrow \lambda(\lambda-3) - 1(\lambda-3) = 0$$

$$\Rightarrow (\lambda-1)(\lambda-3) = 0$$

$\lambda = 1$ and $\lambda = 3$ are eigenvalues. The eigenvectors are $\mathbf{x}$ such that

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x} = \lambda \mathbf{x}$$

For $\lambda = 1$:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - \frac{x_1}{2} \end{bmatrix}$$

$$\Rightarrow 3x_2 = 2x_2 - x_1 \Rightarrow x_2 = -x_1$$

And $(1, -1)$ is an eigenvector for eigenvalue $\lambda = 1$.

For $\lambda = 3$:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 3x_2 - \frac{3x_1}{2} \end{bmatrix}$$

$$\Rightarrow 3x_2 = 6x_2 - 3x_1 \Rightarrow x_2 = x_1$$

And $(1, 1)$ is an eigenvector for eigenvalue $\lambda = 1$.

b.

$$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$$

Answer:

$$\begin{vmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)(1-\lambda) + 2 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 6 = 0 \Rightarrow (\lambda - 3)(\lambda - 2) = 0$$

$\lambda = 2$ and $\lambda = 3$ are eigenvalues. For the eigenvectors:

$$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \mathbf{x} = \lambda \mathbf{x}$$

For $\lambda = 3$:

$$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 \\ 0 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 - \frac{3x_1}{4} \end{bmatrix}$$

$$\Rightarrow \frac{3x_2}{2} = 3x_2 - \frac{3x_1}{4} \Rightarrow \frac{3x_2}{2} = \frac{3x_1}{4} \Rightarrow x_1 = 2x_2$$

$(2, 1)$ is an eigenvector with eigenvalue $\lambda = 3$.

For $\lambda = 2$:

$$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 \\ 0 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 - \frac{2x_1}{4} \end{bmatrix}$$

$$\Rightarrow \frac{3x_2}{2} = 2x_2 - \frac{2x_1}{4} \Rightarrow \frac{x_2}{2} = \frac{2x_1}{4} \Rightarrow x_1 = x_2$$

$(1, 1)$ is an eigenvector with eigenvalue $\lambda = 2$.

References

[1] Wikipedia. http://en.wikipedia.org/wiki/Vector_space

[2] Kenneth Hoffman and Ray Kunze. *Linear Algebra*. Prentice-Hall, 1971.