Determinants and Matrix Inverses: Notes for CSci 124

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Consider the set of n linear equations in n unknowns $(x_1, x_2, ...x_n)$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ . &\quad . \\ a_{n1}x_1 + a_{n2}x_2 + ... + a_{nn}x_n &= b_m \end{aligned}$$

They may be written as a matrix equation:

$$\mathbf{Ax} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1i} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2i} & ... & a_{2n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & ... & a_{ni} & ... & a_{nn} \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix}$$

and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$$

For example, for the set of equations:

$$\begin{aligned} 3u + 5v + w &= 4 \\ 6u + 11v + 4w &= 1 \\ -3u + v - 2w &= 6 \end{aligned}$$

$$\mathbf{A} = \begin{bmatrix} 3 & 5 & 1 \\ 6 & 11 & 4 \\ -3 & 1 & -2 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}$$

and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

One way of solving the equations would be to multiply each side by the inverse of the matrix $\mathbf{A}$:

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} \Rightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

While gaussian elimination, in general, is more efficient than finding $\mathbf{A}^{-1}$ (because you also have to multiply $\mathbf{A}^{-1}$ with $\mathbf{b}$), it is sometimes useful to determine $\mathbf{A}^{-1}$ if one expects to solve several sets of equations with the same value of $\mathbf{A}$, as in the examples discussed in class (computerized tomography and image deblurring). An efficient way of doing so is to use a technique similar to gaussian elimination, known as the Gauss-Jordan method.

1 Gauss-Jordan Method for Matrix Inverse

The matrix to be inverted, $\mathbf{A}$, is written along side the identity matrix of size n, denoted $\mathbf{I}_n$, and gaussian elimination applied:

$$[\mathbf{A}\mathbf{I}_n]$$

Consider the example above:

$$\mathbf{A} = \begin{bmatrix} 3 & 5 & 1 \\ 6 & 11 & 4 \\ -3 & 1 & -2 \end{bmatrix}$$

The matrix is:

$$\begin{bmatrix} 3 & 5 & 1 & 1 & 0 & 0 \\ 6 & 11 & 4 & 0 & 1 & 0 \\ -3 & 1 & -2 & 0 & 0 & 1 \end{bmatrix}$$

and is reduced to:

$$\begin{bmatrix} 3 & 5 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 6 & -1 & 1 & 0 & 1 \end{bmatrix}$$

and then to:

$$\begin{bmatrix} 3 & 5 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -13 & 13 & -6 & 1 \end{bmatrix}$$

Thus the numbers below the diagonal of the matrix

$$\begin{bmatrix} 3 & 5 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -13 \end{bmatrix}$$

are now zero. Now we work on the numbers *above* the diagonal, working from the last row up.

$$\begin{bmatrix} 3 & 5 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -13 & 13 & -6 & 1 \end{bmatrix}$$

becomes:

$$\begin{bmatrix} 3 & 5 & 0 & 2 & \frac{-6}{13} & \frac{1}{13} \\ 0 & 1 & 0 & 0 & \frac{1}{13} & \frac{2}{13} \\ 0 & 0 & -13 & 13 & -6 & 1 \end{bmatrix}$$

and:

$$\begin{bmatrix} 3 & 0 & 0 & 2 & \frac{-11}{13} & \frac{-9}{13} \\ 0 & 1 & 0 & 0 & \frac{1}{13} & \frac{2}{13} \\ 0 & 0 & -13 & 13 & -6 & 1 \end{bmatrix}$$

Now we divide each row so as to get the identity in the first three columns, the other three provide the inverse.

$$\begin{bmatrix} 1 & 0 & 0 & \frac{2}{3} & \frac{-11}{39} & \frac{-9}{39} \\ 0 & 1 & 0 & 0 & \frac{1}{13} & \frac{2}{13} \\ 0 & 0 & 1 & -1 & \frac{6}{13} & -\frac{1}{13} \end{bmatrix}$$

Thus,

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{-11}{39} & \frac{-9}{39} \\ 0 & \frac{1}{13} & \frac{2}{13} \\ -1 & \frac{6}{13} & -\frac{1}{13} \end{bmatrix}$$

And it can be checked that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_3$, and also that $\mathbf{A}^{-1}\mathbf{b} = [11 - 4]$, the solution also obtained using gaussian elimination.

An explicit solution for a set of linear equations can be determined without resorting to gaussian elimination or the Gauss-Jordan inverse, though it is a far less efficient method. It is known as *Cramer's Rule* and uses the notion of the *determinant* of a matrix. Before explaining Cramer's rule, we explain the determinant.

2 Determinants

We start with examples. The determinant of a matrix of size 2×2 is simple:

$$\left| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right| = a_{11}a_{22} - a_{12}a21$$

where $|A|$ denotes the determinant of A, also often denoted $Det(A)$. The determinant may be thought of as the volume of a parallelopiped (solid) whose sides are the column vectors of the matrix. In this case, as the matrix is square matrix of size 2, the volume is the area of the parallelogram defined by the two columns.

Example 1. Find the determinants of the matrices:

a.

$$\left| \begin{bmatrix} 3 & 4 \\ 5 & 9 \end{bmatrix} \right| = 27 - 20 = 7$$

b.

$$\left| \begin{bmatrix} 5 & 4 \\ -5 & 4 \end{bmatrix} \right| = 20 + 20 = 40$$

Now consider a matrix of size 3×3:

$$\left| \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \right| = a_{11} \left| \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \right| - a_{12} \left| \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} \right| + a_{13} \left| \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \right|$$

Here, $\left| \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} \right|$ is termed the *minor* of a_{11}. Notice that a_{12} is multiplied by a negative sign.

Example 2 Find the determinants of the matrices:

a.

$$\left| \begin{bmatrix} 3 & 4 & 1 \\ 5 & 9 & 1 \\ 4 & 2 & 2 \end{bmatrix} \right| = 3(18 - 2) - 4(10 - 4) + 1(10 - 36) = 48 - 24 - 26 = -2$$

b.

$$\left|\begin{bmatrix} 5 & 4 & -1 \\ -5 & 4 & -2 \\ 3 & 2 & 1 \end{bmatrix}\right| = 5(4+4) - 4(-5+6) - 1(-10-12) = 40 - 4 + 22 = 58$$

We now define the determinant of a square matrix of size $n \times n$ (determinants are not defined for matrices that are not square):

$$\left|\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}\right|$$

$$= a_{11}Minor(a_{11}) - a_{12}Minor(a_{12}) + a_{13}Minor(a_{13}) - a_{14}Minor(a_{14}) + \dots (-1)^{1+n} a_{nn}Minor(a_{nn})$$

$$= \sum_{i=1}^{n} (-1)^{i+1} a_{1i} Minor(a_{1i})$$

where $Minor(a_{ij})$ is the determinant of the matrix formed by deleting the row and column containing a_{ij}:

$$Minor(a_{ij}) = \left|\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1,j-1} & a_{1,j+1} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2,j-1} & a_{2,j+1} & \dots & a_{2n} \\ . & . & . & . & & & \\ a_{i-1,1} & a_{i-1,2} & \dots & a_{i-1,j-1} & a_{i-1,j+1} & \dots & a_{i-1,n} \\ a_{i+1,1} & a_{i+1,2} & \dots & a_{i+1,j-1} & a_{i+1,j+1} & \dots & a_{i+1,n} \\ . & . & . & . & & & \\ . & . & . & . & & & \\ a_{n1} & a_{n2} & \dots & a_{n,j-1} & a_{n,j+1} & \dots & a_{nn} \end{bmatrix}\right|$$

Note that a matrix is invertible if and only if its determinant is non-zero.

3 Cramer's Rule for the Solution of Linear Equations

Recall our example for gaussian elimination:

$$\begin{aligned} 3u + 5v + w &= 4 - - - - (1) \\ 6u + 11v + 4w &= 1 - - - - (2) \\ -3u + v - 2w &= 6 - - - - (3) \end{aligned}$$

Instead of using gaussian elimination, we use Cramer's rule to solve these. Cramer's rule applied here is:

$$u = \frac{\begin{vmatrix} 4 & 5 & 1 \\ 1 & 11 & 4 \\ 6 & 1 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & 5 & 1 \\ 6 & 11 & 4 \\ -3 & 1 & -2 \end{vmatrix}} = \frac{4(-22-4) - 5(-2-24) + 1(1-66)}{3(-22-4) - 5(-12+12) + 1(6+33)} = \frac{-104+130-65}{-78+39} = \frac{-39}{-39} = 1$$

$$v = \frac{\begin{vmatrix} 3 & 4 & 1 \\ 6 & 1 & 4 \\ -3 & 6 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & 5 & 1 \\ 6 & 11 & 4 \\ -3 & 1 & -2 \end{vmatrix}} = \frac{3(-2-24) - 4(-12+12) + 1(36+3)}{-39} = \frac{-78+39}{-39} = \frac{-39}{-39} = 1$$

$$w = \frac{\begin{vmatrix} 3 & 5 & 4 \\ 6 & 11 & 1 \\ -3 & 1 & 6 \end{vmatrix}}{\begin{vmatrix} 3 & 5 & 1 \\ 6 & 11 & 4 \\ -3 & 1 & -2 \end{vmatrix}} = \frac{3(66-1) - 5(36+3) + 4(6+33)}{-39} = \frac{195-195+156}{-39} = \frac{156}{-39} = -4$$

In general, for the matrix equations $\mathbf{Ax} = \mathbf{b}$ as defined at the beginning of this set of notes, Cramer's rule provides the following solutions when $Det(\mathbf{A}) \neq 0$, and no solutions otherwise. Thus it only provides a solution when a unique one exists. A unique solution for a matrix equation of the form $\mathbf{AX} = b$ exists if and only if $Det(\mathbf{A}) \neq 0$.

$$x_i = \frac{Det(\mathbf{A_i})}{Det(\mathbf{A})}$$

where $\mathbf{A}$ is as defined earlier, and $\mathbf{A_i}$ is $\mathbf{A}$ with the i^{th} column replaced with the column of the rhs of the equation

$$\mathbf{A_i} = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1,i-1} & b_1 & a_{1,i+1} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2,i-1} & b_2 & a_{2,i+1} & ... & a_{2n} \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ a_{n1} & a_{n2} & ... & a_{n,i-1} & b_n & a_{n,i+1} & ... & a_{nn} \end{bmatrix}$$

4 Exercises

1. Evaluate the determinants of the following matrices:

a.

$$\begin{bmatrix} 2 & 9 \\ 4 & 18 \end{bmatrix}$$

b.

$$\begin{bmatrix} 3 & -2 \\ 7 & 5 \end{bmatrix}$$

c.

$$\begin{bmatrix} 3 & 2 & 5 \\ 4 & 3 & 1 \\ 2 & -3 & -6 \end{bmatrix}$$

d.

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ -1 & -1 & 2 \end{bmatrix}$$

2. Use Cramer's rule to determine the unique solution, if one exists, to the following sets of linear equations.

a.

$$\begin{array}{rcrcr} x & + & 3y & = & 4 \\ 2x & + & y & = & 16 \end{array}$$

b.

$$\begin{array}{rcrcrcr} x & + & y & - & z & = & 4 \\ 2x & + & y & + & z & = & 16 \\ -x & - & y & + & 2z & = & 0 \end{array}$$

(You can use your answer to 1d here).

(the following are the same equations you solved using gaussian elimination):

c.

$$\begin{array}{rcrcrcr} 2u & + & 6v & - & 4w & = & 18 \\ 3u & + & 10v & - & 9w & = & 27 \\ 5u & - & 10v & + & 10w & = & -10 \end{array}$$

d.

$$\begin{array}{rcrcrcr} 3u & + & 8v & + & 4w & = & 7 \\ 3u & + & 10v & - & 2w & = & -3 \\ 6u & - & 10v & + & 10w & = & -8 \end{array}$$

e.

$$\begin{array}{rcrcrcr} u & + & v & + & 2w & = & 2 \\ 5u & + & 10v & + & 9w & = & 4 \\ 2u & - & 8v & + & 3w & = & 13 \end{array}$$

3. Use the Gauss-Jordan method to determine the inverse of the following matrices:

a.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & -2 \end{bmatrix}$$

b.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 0 \\ -1 & 1 & -2 \end{bmatrix}$$

c.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

You can also use Gauss-Jordan to solve the equations in 2 and compare results.

5 Solutions

1. Evaluate the determinants of the following matrices:

a.

$$\begin{bmatrix} 2 & 9 \\ 4 & 18 \end{bmatrix}$$

$2 \times 18 - 9 \times 4 = 0$

b.

$$\begin{bmatrix} 3 & -2 \\ 7 & 5 \end{bmatrix}$$

$3 \times 5 - (-2) \times 7 = 29.$

c.

$$\begin{bmatrix} 3 & 2 & 5 \\ 4 & 3 & 1 \\ 2 & -3 & -6 \end{bmatrix}$$

$3(-18+3) - 2(-26) + 5(-18) = -45 + 52 - 90 = -83$

d.

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ -1 & -1 & 2 \end{bmatrix}$$

(2+1) - (4+1) -(-2 +1) = -1

2. Use Cramer's rule to determine the unique solution, if one exists, to the following sets of linear equations.

a.

$$\begin{array}{rcrcr} x & + & 3y & = & 4 \\ 2x & + & y & = & 16 \end{array}$$

$$x = \frac{\begin{vmatrix} 4 & 3 \\ 16 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{-44}{-5} == \frac{44}{5}$$

$$y = \frac{\begin{vmatrix} 1 & 4 \\ 2 & 16 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{8}{-5} = -\frac{8}{5}$$

Solution is: $x = \frac{44}{5}$, $y = -\frac{8}{5}$

b.

$$\begin{array}{rcrcrcr} x & + & y & - & z & = & 4 \\ 2x & + & y & + & z & = & 16 \\ -x & - & y & + & 2z & = & 0 \end{array}$$

(You can use your answer to 1d here).

$$x = \frac{\begin{vmatrix} 4 & 1 & -1 \\ 16 & 1 & 1 \\ 0 & -1 & 2 \end{vmatrix}}{-1} = \frac{4(2+1) - 1(32-0) - 1(-16-0)}{-1} = 4$$

Denominator is the answer from 1d.

$$y = \frac{\begin{vmatrix} 1 & 4 & -1 \\ 2 & 16 & 1 \\ -1 & 0 & 2 \end{vmatrix}}{-1} = \frac{1(32-0) - 4(4+1) - 1(0+16)}{-1} = 4$$

$$z = \frac{\begin{vmatrix} 1 & 1 & 4 \\ 2 & 1 & 16 \\ -1 & -1 & 0 \end{vmatrix}}{-1} = \frac{1(0+16) - 1(0+16) + 4(-2+1)}{-1} = 4$$

Solution is $x = y = z = 4$

c. Solution is $u = 2$, $v = 3$, $w = 1$.

d. $u = -3$, $v = 1$ and $w = 2$ is the solution.

e. $u = 1$, $v = -1$, and $w = 1$ is the solution.

3. Use the Gauss-Jordan method to determine the inverse of the following matrices:

a.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -1 & 1 & -2 \end{bmatrix}$$

$$[\mathbf{A}|\mathbf{I}_3] = \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & -2 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -3 & 2 & -2 & 1 & 0 \\ 0 & 3 & -3 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -3 & 2 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 & -1 & -1 \\ 0 & -3 & 0 & -4 & 3 & 2 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{-2}{3} & 1 & \frac{1}{3} \\ 0 & -3 & 0 & -4 & 3 & 2 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{-2}{3} & 1 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{4}{3} & -1 & \frac{-2}{3} \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} \frac{-2}{3} & 1 & \frac{1}{3} \\ \frac{4}{3} & -1 & \frac{-2}{3} \\ 1 & -1 & -1 \end{bmatrix}$$

b.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 0 \\ -1 & 1 & -2 \end{bmatrix}$$

$$[\mathbf{A}|\mathbf{I}_3] = \begin{bmatrix} 1 & 2 & -2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & -2 & 0 & 0 & 1 \end{bmatrix}$$

$$[\mathbf{A}|\mathbf{I}_3] = \begin{bmatrix} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 \\ 0 & 3 & -4 & 1 & 0 & 1 \end{bmatrix}$$

$$[\mathbf{A}|\mathbf{I}_3] = \begin{bmatrix} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 2 & -2 & 3 & 1 \end{bmatrix}$$

$$[\mathbf{A}|\mathbf{I}_3] = \begin{bmatrix} 1 & 2 & 0 & -1 & 3 & 1 \\ 0 & -1 & 0 & 1 & -2 & -1 \\ 0 & 0 & 2 & -2 & 3 & 1 \end{bmatrix}$$

$$[\mathbf{A}|\mathbf{I}_3] = \begin{bmatrix} 1 & 0 & 0 & 1 & -3 & -1 \\ 0 & -1 & 0 & 1 & -2 & -1 \\ 0 & 0 & 2 & -2 & 3 & 1 \end{bmatrix}$$

$$[\mathbf{A}|\mathbf{I}_3] = \begin{bmatrix} 1 & 0 & 0 & 1 & -3 & -1 \\ 0 & 1 & 0 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -3 & -1 \\ -1 & 2 & 1 \\ -1 & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

c.

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

$$[\mathbf{A}|\mathbf{I}_3] = \begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$