Complex Numbers: Notes for CSci 124

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1 Motivation for the Definition of Complex Numbers

Suppose you wished to be able to count objects. You would start with the notion of one object, and then add another and so on. That is, you would want a set of numbers that contains the number 1 and is closed with respect to addition. This is the set of *natural numbers*

$$\mathcal{N} = \{1, 2, 3, ...\}$$

Notice that $\mathcal{N}$, is also closed under both addition and multiplication. That is,

$$x \in \mathcal{N} \; and \; y \in \mathcal{N} \Rightarrow xy \in \mathcal{N} \; and \; x + y \in \mathcal{N}$$

However, the converse is not true. That is, $x \in \mathcal{N} \; and \; y \in \mathcal{N}$ does not imply that $x - y \in \mathcal{N}$.

Example 1 Which of the following result in natural numbers? $5 - 3$; $10 - 20$; $7 - 1$; $29 - 30$?

$5 - 3 = 2$ is a natural number; $10 - 20 = -10$ is not; $7 - 1 = 6$ is; $29 - 30 = -1$ is not.

If, in addition to adding objects, you wish to also take away objects from an existing set, you need a set of numbers that includes $\mathcal{N}$ and is closed under subtraction. The smallest such set is the set of *integers*,

$$\mathcal{I} = \{..., -2, -1, 0, 1, 2, 3, ...\}$$

$\mathcal{I}$ is also said to be the *closure* of $\mathcal{N}$ under subtraction. The integers allow us to add and subtract objects, as well as to compare values – such as elevations and temperatures – with a predetermined zero value. However, just as $\mathcal{N}$ is not closed under subtraction, $\mathcal{I}$ is not closed under division: $x \in \mathcal{I} \; and \; y \in \mathcal{I}, y \neq 0$ does not imply that $\frac{x}{y} \in \mathcal{I}$.

Example 2: Which of the following result in integers? $3 - 2$; $4 - 5$; $\frac{5}{2}$; $\frac{25}{5}$; $\frac{-10}{-2}$?

$3 - 2 = 1$ is an integer; as are: $4 - 5 = -1$; $\frac{25}{5} = 5$; $\frac{-10}{-2} = 5$. $\frac{5}{2}$ is not an integer.

The closure of $\mathcal{I}$ under division (except by zero) is the set of *rational numbers*,

$$\mathcal{Q} = \{\frac{p}{q} \mid p, q, \in \mathcal{I}, q \neq 0\}$$

The rational numbers allow us to share objects into equal parts such that each part is not necessarily a whole. For example, 3 apples among 2 children.

1.1 Some properties of $\mathcal{Q}$

Now we see that $\mathcal{Q}$ satisfies the following properties with respect to both addition and multiplication. Let $\circ$ represent either addition or multiplication. Then:

1. *Closure under operation:* If $x, y \in \mathcal{Q}$, $x \circ y \in \mathcal{Q}$.
2. *Identity:* There is an element e, called the identity,such that $x \circ e = x \; \forall \; x$. For addition, $e = 0$, and for multiplication, $e = 1$.
3. *Inverse:* $\forall \; x$ except $x = 0$ when $\circ$ is multiplication, $\exists \; Inv(x)$ such that $x \circ Inv(x) = e$. For addition, $Inv(x) = -x$, and for multiplication, $Inv(x) = x^{-1}$.
4. *Closure under inverse:* $\mathcal{Q}$ is closed under the inverse operation when $\circ$ is addition, and $\mathcal{Q} \backslash \{0\}$ is closed under the inverse operation when $\circ$ is multiplication. (Here $\backslash$ denotes set subtraction.)

$\mathcal{Q}$ is, however, not closed with respect to square roots. That is, $x^2 = y \in \mathcal{Q}$ does not imply that $x \in \mathcal{Q}$. A physical implication of this is that all squares with rational values of area do not have rational values for side lengths.

For example, the square root of 2 cannot be written in the form $\frac{p}{q}$ for integers p and q. Yet, one can see that $\sqrt{2}$ should lie between 1 (the positive square root of 1) and 2 (the positive square root of 4), and it physically represents the side of a square of area 2.

Example 3: Which of the following equations have at least one rational solution?
$x^2 + 1 = 0$; $x^2 - 4 = 0$, $x^2 - \frac{9}{16} = 0$, $x^2 - \frac{2}{3} = 0$.

$x^2 - 4 = 0$ has rational solutions $x = 2$ and $x = -2$; $x^2 - \frac{9}{16} = 0$ has rational solutions $x = \frac{3}{4}$ and $x = \frac{-3}{4}$. The other two equations do not have any rational solutions.

The set of numbers represented by the number line, the real numbers $\mathcal{R}$, contains the square roots of all the natural (and positive rational) numbers. Thus, in particular, it contains all solutions to equations of the form $x^2 - a = 0$ where $a \in \mathcal{Q}$ and $a > 0$. $\mathcal{R}$ does not, however, contain any solutions to the equation $x^3 - 2 = 0$, or to $x^2 + 1 = 0$. While the motivation for studying the roots of such equations is not as obvious as the motivation for studying, say, the rational numbers, we will see that these roots are, indeed, important, when we study the mathematics required for processing audio and video.

Example 4: Which of the following equations have at least one real solution?
$x^2 + 1 = 0$; $x^2 - 4 = 0$, $x^2 - \frac{9}{16} = 0$, $x^2 - \frac{2}{3} = 0$.

All except $x^2 + 1 = 0$.

1.2 Complex Numbers

It has been shown that the set of complex numbers,

$$\mathcal{C} = \{x + iy \mid x, y, \in \mathcal{R},\ and\, i^2 + 1 = 0\}$$

contains the roots of all polynomials with real or complex coefficients. x is the *real part* of complex number $z = x + iy$, and y the *imaginary part*. They are denoted $Re(z)$ and $Im(z)$ respectively. The number iy is *imaginary*, while x is real. When $y = 0$, the complex number is a real number, that is, $\mathcal{R} \subset \mathcal{C}$. $\mathcal{C}$ is represented by the plane, and the complex number $x + iy$ corresponds to the point (x, y) in this plane (see Figure 1). In this plane, the number line, a common representation of $\mathcal{R}$, corresponds to the x axis.

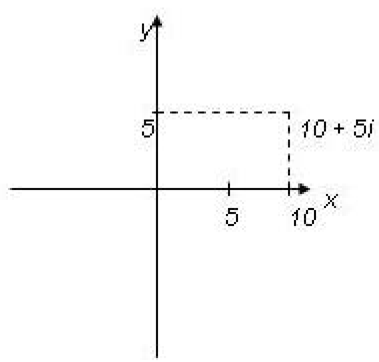

Figure 1: The complex number 10 + 5i on the complex plane

2 Operations on Complex Numbers

All operations on complex numbers are as those on real numbers, with the added condition that $i^2 = -1$.

1. *Equality:* $(x_1 + iy_1) = (x_2 + iy_2) \Rightarrow x_1 = x_2 \; and \; y_1 = y_2$
2. *Addition:* $(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$
3. *Subtraction:* $(x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$
4. *Multiplication:* $(x_1 + iy_1) \times (x_2 + iy_2) = x_1x_2 + iy_1x_2 + ix_1y_2 + i^2y_1y_2 = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)$

Division is slightly more complicated. A small trick helps. Notice that $(x+iy)\times(x-iy) = x^2+y^2$ is real. In fact, it is the *magnitude* of $x + iy$, and $x - iy$ is the *complex conjugate* of $x + iy$. Now,

$$\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1) \times (x_2 - iy_2)}{(x_2 + iy_2) \times (x_2 - iy_2)} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}$$

The *multiplicative inverse* may be similarly determined. Do in class.

Example 5:

A. Determine the following

1. $(2 + 3i) + (5 - 4i)$
2. $(3 + i) - (2 - i)$
3. $(5 - 6i) \times (3 - 2i)$
4. $(1 + 5i)^{-1}$
5. i^{-1}
6. $\frac{4+i}{3-i}$

B. Determine the roots of the quadratic equation $x^2 + 2x + 2 = 0$.

Answers:

A.

1. $(2 + 3i) + (5 - 4i) = (2 + 5) + (3 - 4)i = 7 - i$
2. $(3 + i) - (2 - i) = (3 - 2) + i(1 - -1) = 1 + 2i$
3. $(5 - 6i) \times (3 - 2i) = (5 \times 3 - (-6) \times (-2)) + ((-6) \times 3 + 5 \times (-2)) = 3 - 28i$
4. $(1 + 5i)^{-1} = \frac{1-5i}{5^2+1^2} = \frac{1-5i}{26}$
5. $i^{-1} = \frac{0-i}{0^2+1^2} = -i$
6. $\frac{4+i}{3-i} = \frac{4\times3-\{1\times(-(-1))\}+i(\{4\times(-(-1))\}+1\times3)}{3^2+1^2} = \frac{11+7i}{10}$

B. $x^2 + 2x + 3 = 0 \Rightarrow x = \frac{-2\pm\sqrt{2^2-4\times1\times3}}{2\times1} = -1 \pm \sqrt{2}i$.

3 Polar Representation

Some operations are not as easy to do in the Cartesian or Argand representation of complex numbers (i.e., using their x and y coordinates). It is easier, instead, to represent the complex number in terms of its *magnitude* and *phase*, *argument* or *angle* as shown in the diagram. The magnitude is the distance of the point(x, y) from the origin, and the angle is the angle made by the line joining the origin to the point with the x-axis in the anti-clockwise direction. If r is the magnitude and θ the angle, then:

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= tan^{-1}(\frac{y}{x}) \\ x &= rCos\theta \\ y &= rSin\theta \end{aligned}$$

The complex number is represented $r\angle\theta$ or $re^{i\theta}$. See Figure 2

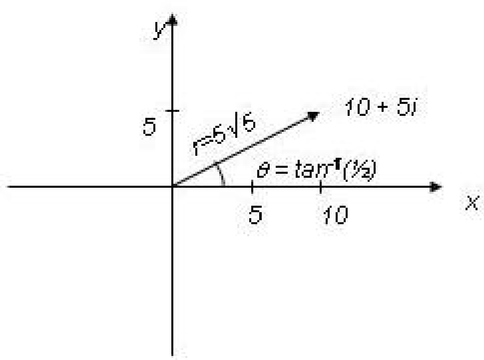

Figure 2: The complex number 10 + 5i in polar form

Example 6:
a. Find the polar representation of the complex number $1 + i$.
b. Find the cartesian representation of the complex number whose magnitude is 2, and angle $30°$.
c. Find the polar representations of $1 + i$, $1 - i$, $-1 + i$, $-1 - i$.
d. Find the polar representations of $\sqrt{3} + i$, $\sqrt{3} - i$, $-\sqrt{3} + i$, $-\sqrt{3} - i$.

Answers:
a. $r = \sqrt{1^2 + 1^2} = \sqrt{2}$
$\theta = tan^{-1}(\frac{1}{1}) = tan^{-1}(1) = 45° = \frac{\pi}{4}$ radians. Hence, $1 + i = \sqrt{2}\angle 45°$ or $\sqrt{2}\angle\frac{\pi}{4}^c$. Also denoted $\sqrt{2}e^{i\frac{\pi}{4}}$.
b. $2e^{i\frac{\pi}{6}} = 2\angle\frac{\pi}{6}^c = 2Cos(\frac{\pi}{6}) + i2Sin(\frac{\pi}{6}) = \sqrt{3} + i$
c. $\sqrt{2}e^{i\frac{\pi}{4}}$, $\sqrt{2}e^{\frac{-i\pi}{4}}$, $\sqrt{2}e^{i\frac{3\pi}{4}}$, $\sqrt{2}e^{i\frac{5\pi}{4}}$.

d. $2e^{i\frac{\pi}{6}}$, $2e^{i\frac{-\pi}{6}}$, $2e^{i\frac{5\pi}{6}}$, $2e^{i\frac{7\pi}{6}}$.

3.1 Multiplication

Recall that

$$(x_1 + iy_1) \times (x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)$$

If $x_1 + iy_1 = r_1e^{i\theta_1}$, and $x_2 + iy_2 = r_2e^{i\theta_2}$, then

$$\begin{aligned} r_1e^{i\theta_1} \times r_2e^{i\theta_2} &= (x_1 + iy_1) \times (x_2 + iy_2) \\ &= r_1r_2(Cos\theta_1Cos\theta_2 - Sin\theta_1Sin\theta_2) + ir_1r_2(Sin\theta_1Cos\theta_2 + Cos\theta_1Sin\theta_2) \\ &= r_1r_2Cos(\theta_1 + \theta_2) + ir_1r_2Sin(\theta_1 + \theta_2) \\ &= r_1r_2e^{i(\theta_1+\theta_2)} \end{aligned}$$

That is, the product of two complex numbers results in a complex number with magnitude the product of the two magnitudes, and angle the sum of the two angles.

Example 7: Using the results of Example 6, find the polar representation of the product of the two numbers $1 + i$ and $\sqrt{3} + 1$. What is the cartesian representation?

Answer: From Example 6, we know that $1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$, and $\sqrt{3} + i = 2e^{i\frac{\pi}{6}}$. Hence

$$(1 + i) \times (\sqrt{3} + i) = 2\sqrt{2}e^{i\frac{5\pi}{12}} = (\sqrt{3} - 1) + i(\sqrt{3} + 1)$$

3.2 Complex Conjugation and Inverse

Theorem: The inverse of a complex number $re^{i\theta}$ is $\frac{1}{r}e^{-i\theta}$.

Proof: Let $z = re^{i\theta}$. Denote its inverse by $z^{-1} = r'e^{i\theta'}$ for some as yet undetermined r' and θ', which we will now determine. Then:

$$z \times z^{-1} = 1 = 1 \times e^{i\times 0} \tag{1}$$

Recall the result proved earlier in class:

$$r_1e^{i\theta_1} \times r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$$

Applying this result to the LHS of (1), we get:

$$z \times z^{-1} = re^{i\theta} \times r'e^{i\theta'} = rr'e^{i(\theta+\theta')}$$

And the above is equal to the RHS of(1), hence:

$$rr'e^{i(\theta+\theta')} = 1 \times e^{i\times 0}$$

Equating magnitudes and angles of the complex numbers on the LHS and the RHS, we get:

$$rr' = 1 \Rightarrow r' = \frac{1}{r}$$

and

$$\theta + \theta' = 0 \Rightarrow \theta' = -\theta$$

This completes the proof.

Theorem If $X = re^{i\theta}$, show that $X^* = re^{-i\theta}$

Example 8: What is the polar representation of $2\sqrt{3} + 2$? Hence, what is the polar representation of its inverse? Its complex conjugate? What are their cartesian representations?

Answer: $r = \sqrt{2^2 \times 3 + 4} = \sqrt{16} = 4$. $tan\theta = \frac{1}{\sqrt{3}}$ and point in the first quadrant, hence $\theta = \frac{\pi}{6}$. The polar representation of $2\sqrt{3} + 2$ is $4e^{i\frac{\pi}{6}}$. The polar representation of its inverse is $\frac{1}{4}e^{\frac{-i\pi}{6}}$, and that of its conjugate: $4e^{\frac{-i\pi}{6}}$. Their cartesian representations are: $\frac{\sqrt{3}}{8} - \frac{1}{8}$ and $2\sqrt{3} + 2$ respectively.

3.3 Powers

Now that we can multiply two complex numbers given their polar representations, we can multiply a number with itself, that is, we can determine the square of a complex number; it is simply another complex number whose magnitude is the square of the original magnitude, and whose angle is twice the angle of the original. Similarly, we can compute any power of a complex number, by induction.

Theorem: $(re^{i\theta})^n = r^n e^{in\theta}$ for $n > 0$, n a natural number.

Proof by induction.

Base Case: The proposition is trivially true for n=0.

Suppose the statement is true for all values of $n \leq k - 1$. We now prove that this implies it is true for $n = k$:

The proposition is true for $n = k - 1$. Hence,

$$\begin{aligned} (re^{i\theta})^{k-1} &= r^{k-1}e^{i(k-1)\theta} \\ \Rightarrow (re^{i\theta})^k &= r^{k-1}e^{i(k-1)\theta} \times re^{i\theta} \\ &= r^k e^{i(k)\theta} \end{aligned}$$

where the last equality follows from the result that $r_1e^{i\theta_1} \times r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}$. Hence the proposition is true for $n = k$. Hence the proposition is true for all finite natural numbers n. This completes the proof.

Example 9: What is the value of $(1 + i\sqrt{3})^4$? $(1 + i\sqrt{3})^6$?

Answer:

$$\begin{aligned} 1 + i\sqrt{3} &= 2e^{i\frac{\pi}{3}} \; polar\ representation \\ \Rightarrow (1 + i\sqrt{3})^4 &= (2)^4 e^{i\frac{4\pi}{3}} \end{aligned}$$

$$\begin{aligned} 1 + i\sqrt{3} &= 2e^{i\frac{\pi}{3}} \\ \Rightarrow (1 + i\sqrt{3})^6 &= (2)^6 e^{i0} \end{aligned}$$

3.4 n^{th} roots of $re^{i\theta}$

Let $z = r'e^{i\theta'}$ be an n^{th} root of $re^{i\theta}$. Then

$$\begin{aligned} z^n = (r'e^{i\theta'})^n &= re^{i\theta} \\ \Rightarrow (r')^n &= r \\ and\ n\theta' &= \theta + m(2\pi) \\ \Rightarrow (r') &= r^{\frac{1}{n}} \\ and\ \theta' &= \frac{\theta}{n} + m\frac{2\pi}{n} \;\; m = 0, 1, ..., n-1 \end{aligned}$$

Example 10: What are the fifth roots of complex number $1 + i\sqrt{3}$?

Answer:

$$\begin{aligned} z^5 = 1 + i\sqrt{3} &= 2e^{i\frac{\pi}{3}} \; polar\ representation \\ \Rightarrow z &= 2^{\frac{1}{5}} e^{i(\frac{\pi}{15} + m\frac{2\pi}{5})} \\ &= 2^{\frac{1}{5}} e^{i\frac{\pi}{15}}, 2^{\frac{1}{5}} e^{i\frac{7\pi}{15}}, 2^{\frac{1}{5}} e^{i\frac{13\pi}{15}}, 2^{\frac{1}{5}} e^{i\frac{4\pi}{15}}, 2^{\frac{1}{5}} e^{i\frac{2\pi}{3}} \end{aligned}$$

3.5 n^{th} roots of unity (the number 1)

We can determine the n^{th} roots of 1 as we would the n^{th} roots of any other complex number: Let $z = r'e^{i\theta'}$ be an n^{th} root of 1. Then

$$\begin{aligned} z^n = (r'e^{i\theta'})^n &= 1 = 1e^{i0} \\ \Rightarrow (r')^n &= 1 \\ and\ n\theta' &= m(2\pi) \\ \Rightarrow (r') &= 1 \end{aligned}$$

$$\begin{aligned} and\ \theta' &= m\frac{2\pi}{n} \ \ m = 0, 1, ..., n-1 \\ &= 0, \frac{2\pi}{n}, \frac{4\pi}{n}, ...\frac{(n-1)2\pi}{n} \end{aligned}$$

Example 11: What are the eighth roots of unity?

Answer:

$$\begin{aligned} z^8 = 1 &= 1 \times e^{i0}\, polar\ representation \\ \Rightarrow z &= 1 \times e^{i\times m\frac{2\pi}{8}} = 1 \times e^{i\times m\frac{\pi}{4}} \\ &= 1, e^{i\frac{\pi}{4}}, e^{i\frac{\pi}{2}}, e^{i\frac{3\pi}{4}}, e^{i\pi}, e^{i\frac{5\pi}{4}}, e^{i\frac{3\pi}{2}}, e^{i\frac{7\pi}{4}} \end{aligned}$$

Note that among the eighth roots of unity are $1, i, -1, -i$.

4 Regions on the Argand Diagram

One can use the Argand diagram to denote values of z that satisfy given equalities and inequalities. See Figures 3 and 4. Figure 3 should be clear. The four inequalities in Figure 4 may be worked through as follows:

$1 < Re(4z) < 3$

Suppose $z = x + iy$.

$$\begin{aligned} 1 < Re(4z) &< 3 \\ \Rightarrow 1 < Re(4x + 4iy) &< 3 \\ \Rightarrow 1 < 4x &< 3 \\ \Rightarrow \frac{1}{4} < x &< \frac{3}{4} \end{aligned}$$

$1 < Magnitude(z^2) < 2$

Suppose $z = re^{i\theta}$.

$$\begin{aligned} 1 < Magnitude(z^2) &< 2 \\ \Rightarrow 1 < Magnitude(r^2 e^{i2\theta}) &< 2 \\ \Rightarrow 1 < r^2 &< 2 \\ \Rightarrow 1 < 3 &< \sqrt{2} \end{aligned}$$

$0 < Argument(z^3) < \frac{\pi}{2}$

Suppose $z = re^{i\theta}$.

$$\begin{aligned}
0 < Argument(z^3) &< \frac{\pi}{2} \\
\Rightarrow 0 < Argument(r^3 e^{i3\theta}) &< \frac{\pi}{2} \\
\Rightarrow 0 < 3\theta + m(2\pi) &< \frac{\pi}{2} \\
\Rightarrow -m\frac{2\pi}{3} < \theta &< \frac{\pi}{6} - m\frac{2\pi}{3} m = 0, 1, 2 \\
\Rightarrow 0 < \theta &< \frac{\pi}{6}(m = 0) \\
OR - \frac{2\pi}{3} = \frac{4\pi}{3} < \theta &< -\frac{\pi}{2} = 3\frac{\pi}{2}(m = 1) \\
OR - \frac{4\pi}{3} = \frac{2\pi}{3} < \theta &< -7\frac{\pi}{6} = 5\frac{\pi}{6}(m = 2)
\end{aligned}$$

$Magnitude(z + 3 - 3i) = 1$

Suppose $z = x + iy$.

$$\begin{aligned}
Magnitude(z + 3 - 3i) &= 1 \\
\Rightarrow Magnitude(x + 3 + (y - 3)i) &= 1 \\
\Rightarrow (x + 3)^2 + (y - 3)^2 &= 1^2 = 1
\end{aligned}$$

is a circle with center at $(-3, 2)$.

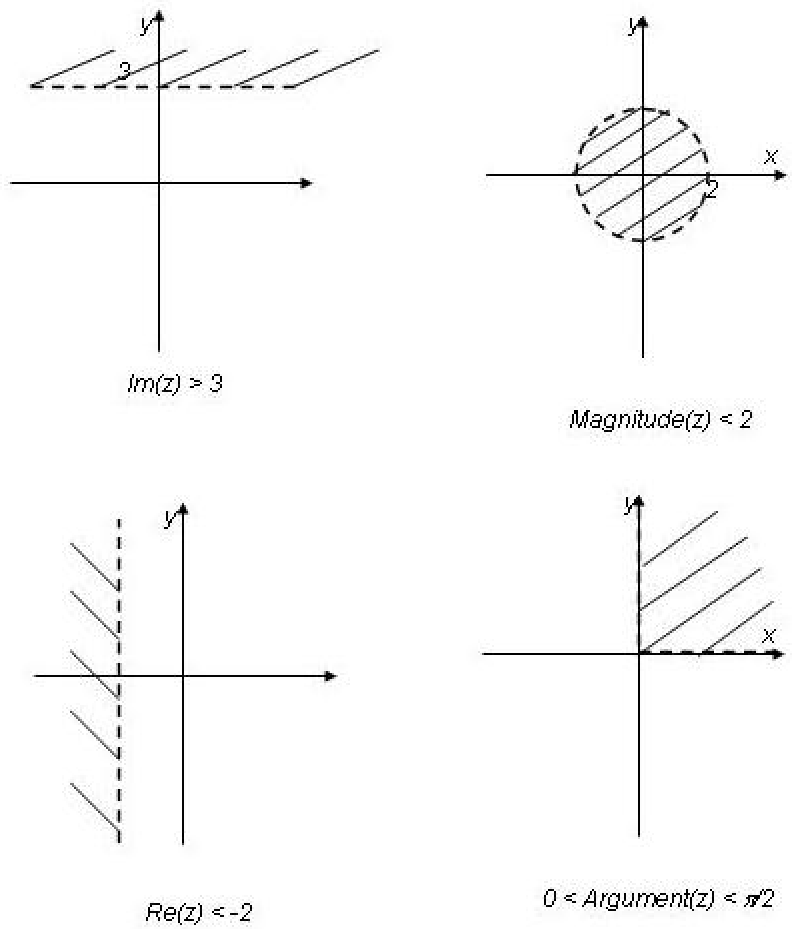

Figure 3: Some inequalities

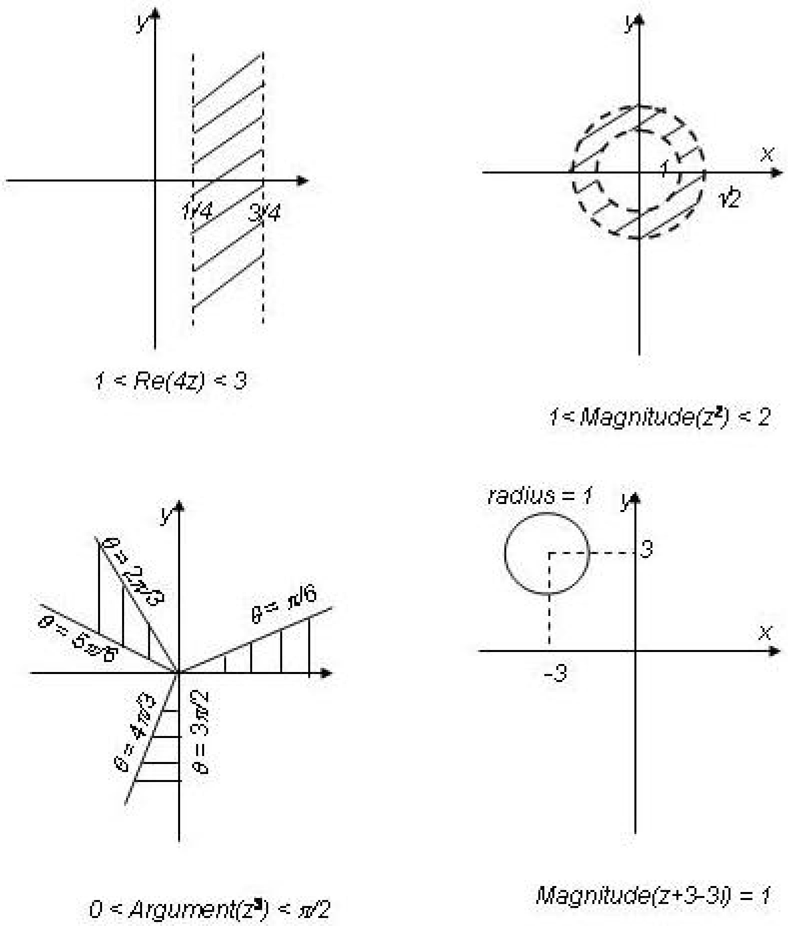

Figure 4: Some inequalities and equalities