CSCI 124 - Discrete Structures II - Spring 2006
George Washington University

Complex Numbers Exercises

1. Evaluate the following:

a. $\frac{(i)^7 \times (3\sqrt{2}+3\sqrt{2}i)^4}{(-6i)^3}$

b. $\frac{7i-1}{1-2i}$

c. $(2\sqrt{3}-1)^2$

d. All fourth roots of -1.

e. All third roots of $1+i$

2. By substituting $z = x + iy$ or $z = re^{i\theta}$ in the following, sketch the regions of the complex plane represented by each equation or inequality as a separate Argand diagram.

a. $Magnitude(z + 1 + 3i) = 2$

b. $\angle 2z = \frac{\pi}{3}$

c. $-2 \leq Re(2iz) \leq 2$

d. $Arg(z^3) = \frac{\pi}{4}$

e. $0 \leq Arg(z^4) \leq \pi$

Solutions

1. Evaluate the following:

$$
\begin{aligned}
\frac{(i)^7 \times (3\sqrt{2}+3\sqrt{2}i)^4}{(-6i)^3} &= \frac{(e^{i\frac{\pi}{2}})^7 \times (6e^{i\frac{\pi}{4}})^4}{(6e^{i\frac{3\pi}{2}})^3} \\
&= \frac{e^{i\frac{7\pi}{2}} \times (6)^4 \times e^{i\pi}}{(6)^3 \times e^{i\frac{9\pi}{2}}} \\
&= 6 \times e^{i(\frac{7\pi}{2}+\pi-\frac{9\pi}{2})} \\
&= 6
\end{aligned}
$$

b.

$$
\begin{aligned}
\frac{7i-1}{1-2i} &= \frac{(7i-1)\times(1+2i)}{(1-2i)\times(1+2i)} \\
&= \frac{(7i-1)\times(1+2i)}{(1-2i)\times(1+2i)} \\
&= \frac{5i-15}{\sqrt{5}} \\
&= \sqrt{5}i - 3\sqrt{5}
\end{aligned}
$$

c. This doesn't contain any complex numbers (no i) – (this was an error on my part)

$$(2\sqrt{3}-1)^2 = 13 - 4\sqrt{3}$$

d. Let $z = re^{i\theta}$ be a fourth root of -1. [Note that the polar form of -1 has magnitude $\sqrt{(-1)^2} = 1$, and angle α such that $Cos(\alpha) = -1$, and $Sin(\alpha) = 0$, that is, $\alpha = \pi$. This can also be determined from the fact that the complex number -1 lies on the negative x axis, and has no imaginary (y) component.] Then:

$$
\begin{aligned}
z^4 &= -1 \\
\Rightarrow r^4 e^{4i\theta} &= 1e^{i\pi} \\
\Rightarrow r^4 &= 1 \; and \\
\Rightarrow 4\theta &= \pi + m2\pi \;\; m = 0,1,2,3
\end{aligned}
$$

$$\begin{aligned} \Rightarrow \theta &= \frac{\pi}{4} + m\frac{\pi}{2} \; m = 0, 1, 2, 3 \\ \Rightarrow z &= e^{\frac{\pi}{4}}, e^{\frac{3\pi}{4}}, e^{\frac{5\pi}{4}}, e^{\frac{7\pi}{4}} \end{aligned}$$

e. Let $z = re^{i\theta}$ be a third root of $1 + i$. [Note that the polar form of $1 + i$ has magnitude $\sqrt{(1)^2 + (1)^2} = \sqrt{2}$, and angle α such that $tan(\alpha) = 1$, and α is in the first quadrant. Hence $\alpha = \frac{\pi}{4}$.] Then:

$$\begin{aligned} z^3 &= 1 + i \\ \Rightarrow r^3 e^{3i\theta} &= \sqrt{2}e^{i\frac{\pi}{4}} \\ \Rightarrow r^3 &= \sqrt{2} \; and \\ \Rightarrow 3\theta &= \frac{\pi}{4} + m2\pi \; m = 0, 1, 2 \\ \Rightarrow r &= (\sqrt{2})^{\frac{1}{3}} = 2^{\frac{1}{6}} \; and \\ \Rightarrow \theta &= \frac{\pi}{12} + m\frac{2\pi}{3} \; m = 0, 1, 2 \\ \Rightarrow z &= 2^{\frac{1}{6}}e^{\frac{\pi}{12}}, 2^{\frac{1}{6}}e^{\frac{3\pi}{4}}, 2^{\frac{1}{6}}e^{\frac{17\pi}{12}} \end{aligned}$$

2. By substituting $z = x + iy$ or $z = re^{i\theta}$ in the following, sketch the regions of the complex plane represented by each equation or inequality as a separate Argand diagram.

See Figure for diagrams.

a. Let $z = x + iy$. Then: $Magnitude(z + 1 + 3i) = 2$

$$\begin{aligned} Magnitude(z + 1 + 3i) &= 2 \\ \Rightarrow (x + 1)^2 + (y + 3)^2 &= 4 \end{aligned}$$

This is a circle about center $(-1, -3)$, with radius 2.

b. Let $z = re^{i\theta}$. Then:

$$\begin{aligned} \angle 2z &= \frac{\pi}{3} \\ \Rightarrow \angle 2re^{i\theta} &= \frac{\pi}{3} \\ \Rightarrow \theta &= \frac{\pi}{3} \end{aligned}$$

c. Let $z = x + iy$. Then:

$$\begin{array}{lrcll} & -2 & \leq & Re(2iz) & \leq & 2 \\ \Rightarrow & -2 & \leq & Re(2ix - 2y) & \leq & 2 \\ \Rightarrow & -2 & \leq & -2y & \leq & 2 \\ \Rightarrow & -1 & \leq & -y & \leq & 1 \\ \Rightarrow & 1 & \geq & y & \geq & -1 \end{array}$$

d. Let $z = re^{i\theta}$. Then:

$$\begin{aligned} Arg(z^3) &= \frac{\pi}{4} \\ \Rightarrow Arg(r^3 e^{i3\theta}) &= \frac{\pi}{4} \\ \Rightarrow 3\theta &= \frac{\pi}{4} + 2m\pi \; m = 0, 1, 2 \\ \Rightarrow \theta &= \frac{\pi}{12} + 2m\frac{\pi}{3} \; m = 0, 1, 2 \\ \Rightarrow \theta &= \frac{\pi}{12}, \frac{3\pi}{4}, \frac{17\pi}{12}, \end{aligned}$$

e. Let $z = re^{i\theta}$. Then:

$$\begin{array}{llclll} & 0 & \leq & Arg(z^4) & \leq & \pi & \\ \Rightarrow & 0 & \leq & Arg(r^4 e^{i4\theta}) & \leq & \pi & \\ \Rightarrow & 0 & \leq & 4\theta + 2m\pi & \leq & \pi & m = 0, 1, 2, 3 \\ \Rightarrow & -m\frac{\pi}{2} & \leq & \theta & \leq & \frac{\pi}{4} - m\frac{\pi}{2} & m = 0, 1, 2, 3 \\ & & & & & & \\ \Rightarrow & 0 & \leq & \theta & \leq & \frac{\pi}{4} & OR \\ & -\frac{\pi}{2} & \leq & \theta & \leq & -\frac{\pi}{4} & OR \\ & -\pi & \leq & \theta & \leq & -\frac{3\pi}{4} & OR \\ & -\frac{3\pi}{2} & \leq & \theta & \leq & -\frac{5\pi}{4} & \end{array}$$

This ends up being easier if the third inequality is written: $2m\pi \leq 4\theta \leq \pi + 2m\pi$, though either is correct:

$$\begin{array}{llclll} & 0 & \leq & Arg(z^4) & \leq & \pi & \\ \Rightarrow & 0 & \leq & Arg(r^4 e^{i4\theta}) & \leq & \pi & \\ \Rightarrow & 2m\pi & \leq & 4\theta & \leq & \pi + 2m\pi & m = 0, 1, 2, 3 \\ \Rightarrow & m\frac{\pi}{2} & \leq & \theta & \leq & \frac{\pi}{4} + m\frac{\pi}{2} & m = 0, 1, 2, 3 \\ & & & & & & \\ \Rightarrow & 0 & \leq & \theta & \leq & \frac{\pi}{4} & OR \\ & \frac{\pi}{2} & \leq & \theta & \leq & \frac{3\pi}{4} & OR \\ & \pi & \leq & \theta & \leq & \frac{5\pi}{4} & OR \\ & \frac{3\pi}{2} & \leq & \theta & \leq & \frac{7\pi}{4} & \end{array}$$

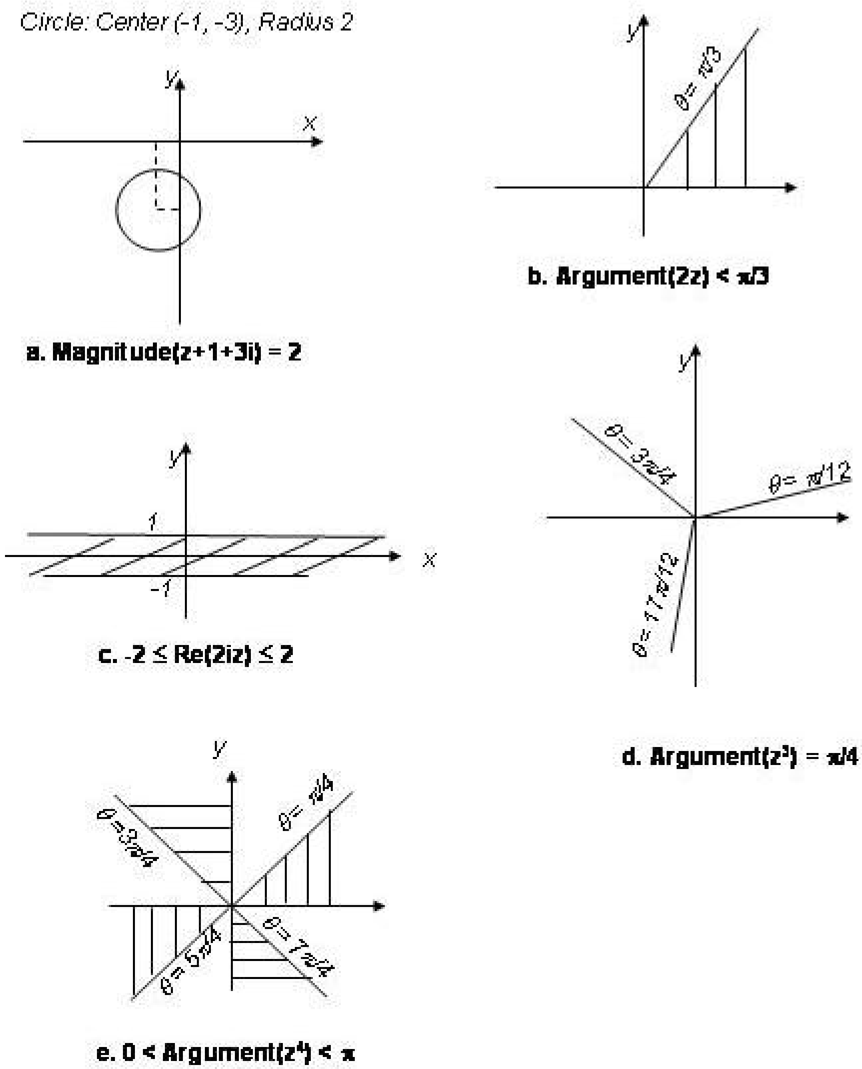

Figure 1: Examples