Vector Spaces and Subspaces: Notes for CSci 124

Poorvi L. Vora

1 Definition

A vector space over a field F (for the moment we consider only the field of real numbers, $\mathbb{R}$) is a set of vectors $\mathbf{V}$ with two operations:

vector addition: $\mathbf{V} \times \mathbf{V} \rightarrow \mathbf{V}$ denoted $\mathbf{v} + \mathbf{w}$ for $\mathbf{v}$ and $\mathbf{w} \in \mathbf{V}$

and

scalar multiplication: $F \times \mathbf{V} \rightarrow \mathbf{V}$ denoted $c\mathbf{v}$ for $c \in F$ and $\mathbf{v} \in \mathbf{V}$

such that:

1. $\mathbf{V}$ is closed with respect to (wrt) vector addition. That is, $\mathbf{v}, \mathbf{w} \in \mathbf{V} \Rightarrow \mathbf{v} + \mathbf{w} \in \mathbf{V}$.
2. $\mathbf{V}$ is closed wrt scalar multiplication. That is, $c \in F, \mathbf{v} \in \mathbf{V} \Rightarrow c\mathbf{v} \in \mathbf{V}$
3. The operation of vector addition has the following properties:
   (a) It is *commutative*. That is, $\mathbf{v}, \mathbf{w} \in \mathbf{V} \Rightarrow \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$
   (b) It is *associative*. That is, $\mathbf{v}, \mathbf{w}, \mathbf{x} \in \mathbf{V} \Rightarrow (\mathbf{v} + \mathbf{w}) + \mathbf{x} = \mathbf{v} + (\mathbf{w} + \mathbf{x})$
   (c) It has an *identity*. That is, $\exists\, \mathbf{e} \in \mathbf{V}$ such that, $\forall\, \mathbf{v} \in \mathbf{V}, \mathbf{v} + \mathbf{e} = \mathbf{v}$.
   $\mathbf{e}$ is often denoted $\mathbf{0}$, the zero vector.
   (d) It has an *inverse*. That is, $\forall\, \mathbf{v} \in \mathbf{V},\ \exists\, (-\mathbf{v}) \in \mathbf{V}$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{e}$.
4. The operation of scalar multiplication has the following properties:
   (a) It has an *identity* that is also the identity of F, denoted 1. That is, $\exists\, 1 \in F$ such that, $\forall\, \mathbf{v} \in \mathbf{V}, 1\mathbf{v} = \mathbf{v}$, and, $\forall\, c \in F, 1c = c$.
   (b) It is consistent with field multiplication. That is, $\mathbf{v} \in \mathbf{V}, c, d, \in F \Rightarrow (cd)\mathbf{v} = c(d\mathbf{v})$
   (c) It is distributive over vector addition. That is, $c \in F, \mathbf{v}, \mathbf{w} \in \mathbf{V} \Rightarrow c(\mathbf{v} + \mathbf{w}) = c\mathbf{v} + c\mathbf{w}$.
   (d) It is distributive over field addition. That is, $\mathbf{v} \in \mathbf{V}, c, d, \in F \Rightarrow (c+d)\mathbf{v} = c\mathbf{v} + d\mathbf{v}$

2 Examples of Vector Spaces

1. The two-dimensional cartesian space over the field of real numbers, $\mathbb{R}$.
2. The three-dimensional cartesian space over $\mathbb{R}$.
3. The n-dimensional cartesian space over $\mathbb{R}$, for positive integer n.
4. $\mathbb{R}$ over $\mathbb{R}$.
5. The set of $n \times m$ matrices for positive integers n and m over $\mathbb{R}$.

3 Examples of the Applications of Ideas from Vector Spaces

Now consider a two-dimensional grey scale image of size $m \times n$, with each pixel taking on values from 0 to 255, inclusive. For example, consider the following image of size 3×4:

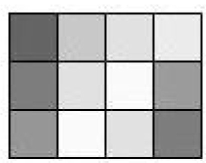

Figure 1: A grey-scale image

It can also be represented as an array of its pixel values:

100	200	225	235
125	225	250	155
150	250	225	125

The grey-scale values are often piled into a single column, row by row. That is, the image is often represented as a single column of size mn, with the first n values being the first row, in order, the next n values the second row, in order, and so on. In our example, the image would be represented as:

$$\begin{bmatrix} 100 \\ 200 \\ 225 \\ 235 \\ 125 \\ 225 \\ 250 \\ 155 \\ 150 \\ 250 \\ 225 \\ 125 \end{bmatrix}$$

One often needs to process these images – for example, to deblur or compress them. That is, one wants to perform mathematical operations on these images. It is hence useful to think of these images as being contained in a vector space of dimension mn (in our example, the dimension is 12) over the real numbers. Addition is defined as regular component-wise addition, and scalar multiplication is defined as multiplication of each pixel value by the scalar.

Now notice that the set of images is, by itself, not a vector space, for several reasons. First, it is not closed wrt field multiplication (take an image that is constant: all pixel values are 200; multiplying it by 2 results in a vector that does not correspond to a grey-scale image, because each component is larger than 255). Second, it is not closed wrt vector addition (take two images, each constant with pixel value 200; adding them results in a vector that does not correspond to a grey-scale image, because each component is larger than 255). Third, the additive inverse of a non-zero image is not among the images, because at least one of its pixel values is negative.

However, one may think of the images as all lying within the vector space during processing. All the processing is performed in the vector space. At the end of the processing, a resulting vector of size mn is obtained. Each of the components of the resulting vector may not be an integer between 0 and 255; that is, the resulting vector may not correspond to a grey-scale image. The resulting vector is transformed into the closest vector that does represent a grey-scale image, and this is the image resulting from the processing.

In our example, we may wish to perform averaging: replace each pixel value with the average pixel value in the 3×3 window around (and including) the pixel. So, for example, the pixel in the second row and second column of our example, of value 225, will be replaced by one of value

$$\frac{100 + 200 + 225 + 125 + 225 + 250 + 150 + 250 + 225}{9} = \frac{1750}{9}$$

rounded off to 194.

Each of the following may also be treated as vectors in a larger vector space:

1. The three color values at a single pixel in a color image.

2. An $n \times m$ color image, which has $3nm$ components (3 colors for each pixel).
3. A video clip consisting of p frames, each a color frame of size $n \times m$. This will have $3pnm$ components (each frame has $3nm$ components).
4. n consecutive samples of a sound signal, taken Δt apart.

4 Example of a Set that is not a Vector Space

Example 1: The first quadrant in two-dimensional euclidean space over the field of real numbers, $\mathbb{R}$.

$$\mathbf{A} = \{\mathbf{b} = (x, y) | x, y \in \mathbb{R}, x \geq 0, y \geq 0\}$$

(The symbol "|" stands for "such that").

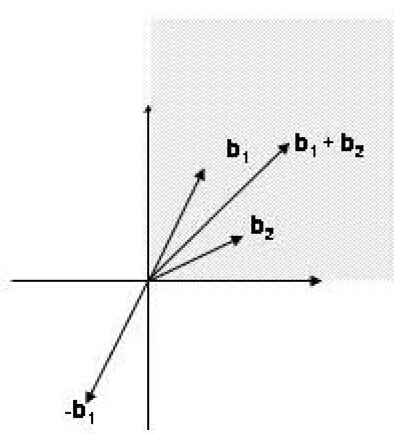

Figure 2: The first quadrant is not a vector space

Is **A** closed wrt vector addition?

Take any two vectors from **A**, (x_1, y_1) and (x_2, y_2). Their sum is $(x_1 + x_2, y_1 + y_2)$. This is in **A** if and only if $x_1 + x_2, y_1 + y_2 \in \mathbb{R}$, and $x_1 + x_2 \geq 0, y_1 + y_2 \geq 0$

- By the definition of **A**, $x_1, y_1, x_2, y_2 \in \mathbb{R}$. This implies that $x_1 + x_2, y_1 + y_2 \in \mathbb{R}$.
- Also by the definition of **A**, $x_1 \geq 0, y_1 \geq 0, x_2 \geq 0, y_2 \geq 0$. This implies that $x_1 + x_2 \geq 0, y_1 + y_2 \geq 0$. (See Figure 2).

Hence **A** is closed wrt vector addition.

Is **A** closed wrt scalar multiplication?

Take any vector (x, y) from $\mathbf{A}$, and any scalar c from $\mathbb{R}$. Then $c(x, y) = (cx, cy)$. This is in $\mathbf{A}$ if and only if $cx, cy \in \mathbb{R}$, and $cx \geq 0, cy \geq 0$.

- By the definition of $\mathbf{A}$, $x, y \in \mathbb{R}$. This implies that $cx, cy \in \mathbb{R}$ (product of two real numbers is real).
- Also by the definition of $\mathbf{A}$, $x \geq 0, y \geq 0$. If $c \geq 0$, then $cx \geq 0$, $cy \geq 0$. However, if $c < 0$, $cx \leq 0$, $cy \leq 0$. In particular, for any $x, y \neq 0$ and $c < 0$, $c(x, y)$ is not in $\mathbf{A}$. For example, if $c = -1$, and $\mathbf{b} \in \mathbf{A}$, but $\mathbf{b} \neq (0, 0)$, $c\mathbf{b}$ is in the third quadrant.

Hence **A** is not closed wrt scalar multiplication.

Hence **A** is not a vector space.

5 Subspaces

Definition A subset **A** of vector space **V** is a *subspace* of **V** if it is closed wrt vector addition and scalar multiplication.

Examples: The x and y axes are subspaces of $\mathbb{R}^2$ as is any straight line passing through the origin.

Example 2: Consider a straight line:

$$\mathbf{A} = \{\mathbf{b} = (x, y) | y = mx + c\}$$

where m and c are real constants.

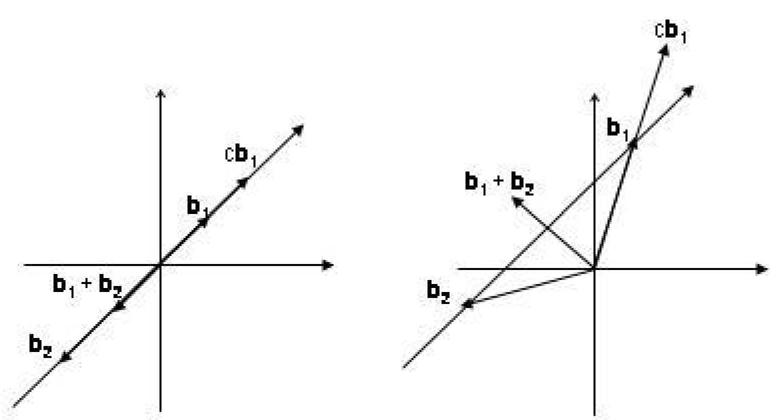

Figure 3: A straight line is a subspace of $\mathbb{R}^2$ if and only if it passes through the origin

Consider any two vectors in $\mathbf{A}$: $\mathbf{b}_1 = (x_1, y_1)$ and $\mathbf{b}_2 = (x_2, y_2)$. Then $\mathbf{b}_1 + \mathbf{b}_2 = (x_1 + x_2, y_1 + y_2)$. Is $\mathbf{b}_1 + \mathbf{b}_2$ in $\mathbf{A}$? For this, we need to determine if $\mathbf{b}_1 + \mathbf{b}_2$ is on the line, that is if

$$y_1 + y_2 = m(x_1 + x_2) + c \tag{1}$$

We know that $y_1 = mx_1 + c$ and $y_2 = mx_2 + c$. Hence

$$y_1 + y_2 = m(x_1 + x_2) + 2c \tag{2}$$

Equation (2) is equation (1) if and only if $c = 0$. That is, $\mathbf{A}$ is closed under vector addition if and only if $c = 0$, i.e. the line passes through the origin. (See Figure 3).

Now consider scalar multiplication. Let $\alpha \in \mathbb{R}$. Let $\mathbf{b} = (x, y) \in \mathbf{V}$. Then $\alpha\mathbf{b} = (\alpha x, \alpha y)$. This is in $\mathbf{A}$ if

$$\alpha y = \alpha m x + c \tag{3}$$

In this case,

$$y = mx + c \Rightarrow \alpha y = \alpha m x + \alpha c \tag{4}$$

Equation (4) is equation (3) if and only if $c = 0$. That is, $\mathbf{A}$ is closed under vector addition if and only if $c = 0$, i.e. the line passes through the origin. (See Figure 3).

Hence any straight line through the origin is a subspace of $\mathbb{R}^2$, but a straight line not through the origin is not.

6 Exercises

1. Consider the union of the first and third quadrants in two-dimensional euclidean space over the field of real numbers, $\mathbb{R}$.

$$\mathbf{A} = \{b = (x, y) | x, y \in \mathbb{R}, x \geq 0, y \geq 0\} \cup \{b = (x, y) | x, y \in \mathbb{R}, x \leq 0, y \leq 0\}$$

Is $\mathbf{A}$ closed wrt vector addition? Is it closed under scalar multiplication? Is it a vector space?

2. Is the set of integers, $\mathbb{Z}$, a vector space over $\mathbb{R}$?