

CSCI 124: Discrete Structures II: Properties of the Inner Product

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Consider the inner product of any two vectors of a vector space V :

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix}$$

Observe that

1. The inner product is symmetric, i.e.:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$$

Proof:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n x_i y_i = \sum_{i=1}^n y_i x_i = \langle \mathbf{y}, \mathbf{x} \rangle$$

2. The inner product is bilinear. That is, if $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$:

a

$$\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$$

and

b

$$\langle \mathbf{x} + \mathbf{z}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle$$

Proof:

First we show (a):

The left-hand side is:

$$\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \sum_{i=1}^n x_i (\mathbf{y} + \mathbf{z})_i = \sum_{i=1}^n x_i (y_i + z_i) = \sum_{i=1}^n x_i y_i + \sum_{i=1}^n x_i z_i = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$$

is the right-hand side.

Similarly we show (b):

The left-hand side is:

$$\langle \mathbf{x} + \mathbf{z}, \mathbf{y} \rangle = \sum_{i=1}^n (\mathbf{x} + \mathbf{z})_i y_i = \sum_{i=1}^n (x_i + z_i) y_i = \sum_{i=1}^n x_i y_i + \sum_{i=1}^n z_i y_i = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle$$

is the right-hand side.

3. Further, observe that:

$$\langle \mathbf{x}, c\mathbf{y} \rangle = c\langle \mathbf{x}, \mathbf{y} \rangle = \langle c\mathbf{x}, \mathbf{y} \rangle = \left\langle \frac{c}{r}\mathbf{x}, r\mathbf{y} \right\rangle \forall c, r \in \mathbb{R}, r \neq 0$$

Proof:

Consider some $c, r \in \mathbb{R}, r \neq 0$. The left hand side is:

$$\langle \mathbf{x}, c\mathbf{y} \rangle = \sum_{i=1}^n x_i(cy_i) = c \sum_{i=1}^n x_i y_i = c\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n (cx_i)y_i = \langle c\mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^n \left(\frac{c}{r}x_i\right)ry_i = \left\langle \frac{c}{r}\mathbf{x}, r\mathbf{y} \right\rangle$$

4. Using 2a and mathematical induction, show in the discussion session that, given k vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ for k a positive integer,

$$\left\langle \mathbf{x}, \sum_{i=1}^k \mathbf{v}_i \right\rangle = \sum_{i=1}^k \langle \mathbf{x}, \mathbf{v}_i \rangle$$

5. Suppose $\{\mathbf{b}_i\}_{i=1}^n$ is a basis for a vector space V and that $\{\mathbf{b}_i\}_{i=1}^n$ is also an orthonormal set. That is, $\{\mathbf{b}_i\}_{i=1}^n$ is an orthonormal basis for a vector space V . Any vector $\mathbf{x} \in V$ can be expressed as

$$\mathbf{x} = \sum_{i=1}^n \langle \mathbf{x}, \mathbf{b}_i \rangle \mathbf{b}_i$$

Proof: Because $\{\mathbf{b}_i\}_{i=1}^n$ is a basis for V , and $\mathbf{x} \in V$, \mathbf{x} can be expressed as a linear combination of $\{\mathbf{b}_i\}_{i=1}^n$. That is, $\exists c_1, c_2, \dots, c_n \in \mathbb{R}$ such that $\mathbf{x} = \sum_{i=1}^n c_i \mathbf{b}_i$. Further,

$$\langle \mathbf{b}_i, \mathbf{x} \rangle = \left\langle \mathbf{b}_i, \sum_{j=1}^n c_j \mathbf{b}_j \right\rangle = \sum_{j=1}^n \langle \mathbf{b}_i, c_j \mathbf{b}_j \rangle = \sum_{j=1}^n c_j \langle \mathbf{b}_i, \mathbf{b}_j \rangle = 0 + 0 + \dots + c_i \langle \mathbf{b}_i, \mathbf{b}_i \rangle + 0 + 0 \dots + 0 = c_i$$

Hence:

$$\mathbf{x} = \sum_{i=1}^n \langle \mathbf{x}, \mathbf{b}_i \rangle \mathbf{b}_i$$

5. Use 4 to find c_1, c_2, c_3, c_4 when $\mathbf{x} = \sum_{i=1}^4 c_i \mathbf{b}_i$ for:

$$\mathbf{x} = \begin{bmatrix} 1 \\ 9 \\ 7 \\ 2 \end{bmatrix} \quad \mathbf{b}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{b}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad \mathbf{b}_3 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} \quad \mathbf{b}_4 = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$