

CSCI 124: Discrete Structures II: Mathematical Induction – Review

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We review mathematical induction, which you should have seen in a first course in discrete mathematics. You should make sure you are *very* familiar with it (do the problems at the back) as we will use it all the time in this course.

The (*weak*) *principle of mathematical induction* may be loosely stated as follows:

If:

- A statement about integer n is true for integer n_0 and
- Truth of the statement for $n = k$ implies truth for $n = k + 1$

then the statement must be true for $n \geq n_0$.

To apply this principle to a particular statement, you would show:

- The basis: show that the statement is true for $n = n_0$
- The inductive step: show that, if the statement is true for $n = k$, then it is true for $n = k + 1$

Examples:

$$1. \sum_{i=1}^n i = \frac{n(n+1)}{2} \quad \forall n \geq 1 \quad (\forall \text{ denotes "for every"})$$

First, identify the desired value, the one for which you seek a formula. That is, the lhs. The desired value is $\sum_{i=1}^n i$. Second, identify the formula you think will work to obtain the desired value, denote this $T(n)$. This is the rhs. $T(n) = \frac{n(n+1)}{2}$. You seek to prove the statement that the desired value = $T(n)$ for all $n \geq 1$.

The basis: the statement is true for $n = 1$, because lhs = 1 and rhs = 1. That is, $T(1)$ does give you the desired value for $n = 1$.

The inductive step. Suppose the statement is true for $n = k$. (This means, suppose you can obtain the desired value for $n = k$ by using the formula $T(k)$). That is, suppose the following is true:

$$\sum_{i=1}^k i = \frac{k(k+1)}{2} \tag{1}$$

We want to determine if the statement is true for $n = k + 1$. That is, we want to see if the desired value can be obtained by using $T(k + 1)$. We have

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^k i + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

by the assumption of truth for $n = k$, (1). This further simplifies to:

$$\sum_{i=1}^{k+1} i = (k+1) \times \left(\frac{k}{2} + 1\right) = \frac{(k+1)(k+2)}{2} = T(k+1)$$

which means that the desired value for $n = k + 1$ is equal to $T(k + 1)$. Hence the statement is also true for $n = k + 1$. Hence, if the statement is true for $n = k$, it is also true for $n = k + 1$.

From the basis and the inductive step, we see that the statement is true for $n \geq 1$.

$$2. n! \leq n^n \quad \forall n \geq 1$$

The desired value is $n!$. $T(n) = n^n$. We wish to show that desired value $\leq T(n) \forall n \geq 1$.

The basis: the statement is true for $n = 1$, because lhs = 1 and rhs = 1, and, hence, desired value $\leq T(1)$.

The inductive step. Suppose the statement is true for $n = k$. That is, suppose the following is true:

$$k! \leq k^k \tag{2}$$

We want to determine if it is true for $n = k + 1$. We have

$$(k + 1)! = (k + 1) \times k! \leq (k + 1) \times k^k$$

by the assumption of truth for $n = k$, (2). This further simplifies to:

$$(k + 1)! \leq (k + 1) \times (k + 1)^k \leq (k + 1)^{k+1} = T(k + 1)$$

which means the statement is also true for $n = k + 1$. Hence, if the statement is true for $n = k$, it is also true for $n = k + 1$.

From the basis and the inductive step, we see that the statement is true for $n \geq 1$.

Practice Examples

Use mathematical induction for the following:

1. Guess a formula for the sum of the first n even integers and prove it.

2. Show that:

a. $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6} \quad \forall n \geq 1$

b. $\sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2}\right]^2 \quad \forall n \geq 1$

c. $\sum_{i=1}^n i(i+1) = \frac{n(n+1)(n+2)}{3} \quad \forall n \geq 1$. Can you show this by using formulas already derived, and without using mathematical induction explicitly?

d. The sum of the first n odd numbers is n^2 . Can you show this by using formulas already derived, and without using mathematical induction explicitly?

3. Guess a formula for $\prod_{i=1}^n 2^i$ and prove it. Can you show this by using formulas already derived, and without using mathematical induction explicitly?

4. Show that:

a. $2^n \leq n! \quad \forall n \geq 4$

b. $n^2 < n! \quad \forall n \geq 4$

c. If $h \geq -1$, then $1 + nh \leq (1 + h)^n \quad \forall n \geq 0$