

CSCI 124/224

Discrete Structures II: Mathematical Induction – Review

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We review mathematical induction, which you should have seen in a first course in discrete mathematics. You should make sure you are *very* familiar with it (do the problems at the back) as we will use it all the time in this course.

The (*weak*) *principle of mathematical induction* may be loosely stated as follows:

If:

- A statement about integer n is true for integer n_0 and
- Truth of the statement for $n = k$ implies truth for $n = k + 1$

then the statement must be true for $n \geq n_0$.

To apply this principle to a particular statement, you would show:

- The basis: show that the statement is true for $n = n_0$
- The inductive step: show that, if the statement is true for $n = k$, then it is true for $n = k + 1$

Examples:

1. $\sum_{i=1}^n i = \frac{n(n+1)}{2} \quad \forall n \geq 1$ (\forall denotes “for every”)

The basis: the statement is true for $n = 1$, because lhs = 1 and rhs = 1.

The inductive step. Suppose the statement is true for $n = k$. That is, suppose the following is true:

$$\sum_{i=1}^k i = \frac{k(k+1)}{2} \tag{1}$$

We want to determine if it is true for $n = k + 1$. We have

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^k i + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

by the assumption of truth for $n = k$, (1). This further simplifies to:

$$\sum_{i=1}^{k+1} i = (k+1) \times \left(\frac{k}{2} + 1\right) = \frac{(k+1)(k+2)}{2}$$

which means the statement is also true for $n = k + 1$. Hence, if the statement is true for $n = k$, it is also true for $n = k + 1$.

From the basis and the inductive step, we see that the statement is true for $n \geq 1$.

$$2. n! \leq n^n \quad \forall n \geq 1$$

The basis: the statement is true for $n = 1$, because lhs = 1 and rhs = 1.

The inductive step. Suppose the statement is true for $n = k$. That is, suppose the following is true:

$$k! \leq k^k \tag{2}$$

We want to determine if it is true for $n = k + 1$. We have

$$(k + 1)! = (k + 1) \times k! \leq (k + 1) \times k^k$$

by the assumption of truth for $n = k$, (2). This further simplifies to:

$$(k + 1)! \leq (k + 1) \times (k + 1)^k \leq (k + 1)^{k+1}$$

which means the statement is also true for $n = k + 1$. Hence, if the statement is true for $n = k$, it is also true for $n = k + 1$.

From the basis and the inductive step, we see that the statement is true for $n \geq 1$.

Practice Examples

Use mathematical induction for the following:

1. Guess a formula for the sum of the first n even integers and prove it.

2. Show that:

a. $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$

b. $\sum_{i=1}^n i^3 = \left[\frac{n(n+1)}{2}\right]^2$

c. $\sum_{i=1}^n i(i+1) = \frac{n(n+1)(n+2)}{3}$. Can you show this by using formulas already derived, and without using mathematical induction explicitly?

d. The sum of the first n odd numbers is n^2 . Can you show this by using formulas already derived, and without using mathematical induction explicitly?

3. Guess a formula for $\prod_{i=1}^n 2^i$ and prove it. Can you show this by using formulas already derived, and without using mathematical induction explicitly?

4. Show that:

a. $2^n \leq n! \quad \forall n \geq 4$

b. $n^2 < n! \quad \forall n \geq 4$

c. If $h \geq -1$, then $1 + nh \leq (1 + h)^n \quad \forall n \geq 0$