

**CSCI 124/224 - Discrete Structures II - Fall 2010**  
**George Washington University**

**Homework 3: 50 points**

due 12 November 2010, by 6 pm in TA's mailbox.

**Policy on collaboration:** All examinations, papers, and other graded work products and assignments are to be completed in conformance with The George Washington University Code of Academic Integrity. You may discuss HWs among yourselves, and work on them in groups. However, each student is expected to write his or her own HW out independently; you may not copy one another's assignments, even in part. You may not collaborate with others on the quizzes, tests or final.

You are expected to cite all your sources in any written work that is not closed book: papers, books, web sites, discussions with others - faculty, friends, students. For example, if, in a group, one student has a major idea that leads to a solution to a HW problem, all other students in the group should cite this student.

*Any violations will be treated as violations of the Code of Academic Integrity.*

- (6 points) Let  $f$  be a homomorphism from group  $G_1$  with operation  $\diamond$  to group  $G_2$  with operation  $\circ$ . Show that  $H$ , the image of  $G_1$  in  $G_2$

$$H = f(G_1) = \{f(g) | g \in G_1\}$$

is a subgroup of  $G_2$ .

(Hint: Use the facts that: for any  $y \in H$ ,  $y = f(x)$  for some  $x \in G_1$ , that  $\forall x \in G_1$ ,  $f(x^{-1}) = (f(x))^{-1}$  and that  $f(e) = \bar{e}$  where  $e$  and  $\bar{e}$  are identities in  $G_1$  and  $G_2$  respectively.)

- Let  $f$  be a homomorphism from group  $G_1$  with operation  $\diamond$  to group  $G_2$  with operation  $\circ$ . Consider the subset of group  $G_1$

$$H_1 = \{x | f(x) = e\}$$

Show that:

- (6 points)  $H_1$  is a subgroup of  $G_1$ .
  - (12 points)  $f$  is an isomorphism if and only if  $H_1 = \{e\}$  where  $e$  is the identity in  $G_1$  wrt  $\diamond$ .
- (8 points) Let  $f$  be an isomorphism from group  $G_1$  with operation  $\diamond$  to group  $G_2$  with operation  $\circ$ . Show that  $f^{-1}$  is also an isomorphism.
  - Consider some set  $G$  with some operation  $\diamond$ . ( $G$  and  $\diamond$  are not necessarily related to any set or operation mentioned earlier in this HW or elsewhere.). Suppose  $G$  is a group with respect to  $\diamond$ , with identity  $e$ . Let  $\bar{z}$  denote the inverse of  $z$  with respect to  $\diamond$ . Answer the following:

- (a) Let  $a$  be a fixed element in  $G$ . Define another operation  $\square$  as follows:

$$\forall x, y \in G, x \square y = x \diamond y \diamond a$$

- i. (1 points) Is  $G$  closed with respect to  $\square$ ?
  - ii. (1 points) Is  $\square$  associative?
  - iii. (1 points) Is  $\square$  associative if  $\diamond$  is commutative? (DO NOT assume  $\diamond$  is commutative for any other part of this problem).
  - iv. (2 points) Does  $\square$  have an identity in  $G$ ? If so, what is it?
  - v. (2 points) Consider  $x$ , some element in  $G$ . Does the inverse of  $x$  with respect to  $\square$  exist in  $G$ ? If so, what is it?
- (b) (5 points) Define another operation  $\blacklozenge$  on  $G$  as follows:

$$\forall x, y \in G, x \blacklozenge y = \overline{x \diamond y}$$

Show that  $G$  is not a group with respect to  $\blacklozenge$ . (Hint: First identify which property it would not satisfy).

5. Provide clear reasoning for the following. There will be no credit without reasoning.

- (a) Let  $f$  be a homomorphism from  $\mathbb{Z}_{11}^*$  with operation multiplication *modulo* 11, to itself. (Recall that  $\mathbb{Z}_p^*$  is the set of integers smaller than  $p$  that are relatively prime to  $p$ ). Suppose  $f(3) = 2$ . What are the values of:
- i. (1 point)  $f(9)$ ?
  - ii. (1 point)  $f(4)$ ?
- (b) Let  $f$  be a homomorphism from  $\mathbb{Z} \times \mathbb{Z}$  with operation component-wise addition, to itself. Suppose  $f((-2, 1)) = (1, 1)$  and  $f((1, 0)) = (0, 1)$ . What are the values of:
- i. (1 point)  $f((-1, 1))$
  - ii. (1 point)  $f((3, 5))$
  - iii. (2 points)  $f((n, m))$